
Volume 93, Issue 7, June 2011
ISSN: 0263-8223
ELSEVIER
COMPOSITE STRUCTURES
EDITOR: A. J. M. FERREIRA

Composite Structures 93 (2011) 1664–1672

Contents lists available at ScienceDirect

Composite Structures

journal homepage: www.elsevier.com/locate/compstruct

# Exact solutions for free vibrations of orthotropic rectangular Mindlin plates

Bo Liu *, Yufeng Xing

*The Solid Mechanics Research Center, Beijing University of Aeronautics and Astronautics, Beijing 100191, China*

ARTICLE INFO

*Article history:*
Available online 22 January 2011

*Keywords:*
Orthotropic
Mindlin plates
Free vibration
Frequency
Separation of variables

ABSTRACT

Exact closed-form solutions are obtained for free vibrations of orthotropic rectangular Mindlin plates by using the separation of variables method although it is difficult to solve them. The plates have two opposite edges simply supported and all possible combinations of classical boundary conditions at the other two edges. The exact solutions of orthotropic rectangular Mindlin plates are compared with those of isotropic ones and their differences are discussed. The exact solutions are validated through both mathematical proof and numerical comparisons with available *p*-Ritz solutions and the differential quadrature finite element method solutions calculated by the authors.



## 1. Introduction

The orthotropic plates are commonly used in aerospace engineering as well as automobile and ship vehicles and considered as the fundamental structural elements [1–3]. The wide use of advanced composite laminates in industrial mainly because they exhibit properties which are more favorable than those of single-layer and isotropic ones. Also, fiber enforced composite materials have high *strength*/*weight* and *stiffness*/*weight* ratios, relatively low cost [4], corrosion resistance and longer fatigue life [5]. The orthotropic behavior arises from the use of materials with such constitutive relations, and many composite plates can also be modeled analytically as orthotropic plates [6]. The ratio of in-plane Young's modulus to transverse shear modulus is relatively high for composite plates due to the great difference between elastic properties of fiber filament and matrix materials. This leads to using the thin plate theory, which neglects transverse shear deformation, is invalid for most composite plates, even those which are geometrically thin [7]. For dealing with complicated shear strain distribution, various shear deformation theories have been proposed (for example, [8–14]). The Mindlin plate theory, generally referred to as the first order shear deformation theory and incorporated the effect of rotary inertial, is one of the most typical and used deformation theories for the analysis of composite laminates [15].

Exact solutions for free vibrations of isotropic rectangular Mindlin plates have been obtained for some classical boundary conditions. Mindlin [9] first uncoupled the equations by expressing the two rotations ($\psi_x, \psi_y$) and the deflection ($w$) in terms of three potentials. Then Mindlin et al. [16] presented exact solutions for simply supported rectangular plates and studied coupling of modes for the case of one pair of parallel edges free and the other pair simply supported. Special attention was given to the high modes and frequencies of vibration which were beyond the range of applicability of the classical theory of thin plates. Later, Brunelle [17] and Shahrokh and Arsanjani [18] derived the exact characteristic equations for elastic stability and free vibration, respectively, of a rectangular Mindlin plate with two parallel edges simply supported and the remaining two edges subjected to a variety of boundary conditions. The exact solutions for free vibrations of Mindlin plates are much more complex than those of thin plates. Recently, Xing and Liu [19,20] presented simplified characteristic equations, which were similar with those via Kirchhoff thin plate, for free vibrating Mindlin plates with any combinations of simply supported and clamped edges, involved free edges for plates with two simply supported opposite edges.

Exact solutions for orthotropic rectangular Mindlin plates, however, are only available for simply supported plates so far [21]. For other classical boundary conditions, the application of an approximate method was regarded as unavoidable [22,4,7]. On this aspect, Liew et al. [23] have presented a comprehensive literature survey on the research works up to 1994 on vibrations of thick plates: 132 publications have been cited, attention has been mainly devoted to studies based on the vibration of thick laminated plates. Apparently the finite element technique [24] and the Rayleigh-Ritz technique [4] have been most widely used in free vibration analysis of orthotropic Mindlin rectangular plates. Other methods such as Galerkin technique [25], the superposition method [26], and the finite difference method [7] etc. have also been used to the free vibration analysis of orthotropic rectangular Mindlin plates. The state space concept has been used to develop Levy-type exact solutions for free vibration and buckling of laminated composite plates based on the first order and higher order theories [27,28].

* Corresponding author.
*E-mail address:* liubo68@buaa.edu.cn (B. Liu).

0263-8223/$ - see front matter © 2011 Elsevier Ltd. All rights reserved.
doi:10.1016/j.compstruct.2011.01.014

However, it is computational expensive and exhibits computational difficulty even if the relative thickness ratio is not small ($t/b \leqslant 0.1$, [29]).

In this context, the present paper is devoted to solve the exact closed-form solutions by using the separation of variables method [3,19,20,30]. The work of this paper includes:

(1) Five types of exact eigensolutions are obtained for the first time for orthotropic rectangular Mindlin plates. Calculation shows that the eigenvalues for such problems are real or pure imaginary, as in the Kirchhoff rectangular thin plates or isotropic rectangular Mindlin plates.
(2) The exact solutions are validated through both mathematical proof and numerical comparisons with available *p*-Ritz solutions [4] and the differential quadrature finite element method [31] solutions calculated by the authors.
(3) It is found that, for orthotropic plates, not only the two rotations ($\psi_x, \psi_y$) but also the deflection ($w$) need to be expressed by the three potentials of eigenfunctions, which is different from isotropic Mindlin plates where the deflection is expressed only via two potentials.

It is well known that, of all the available solutions, the exact solutions which satisfy both the governing equations and the boundary conditions rigorously are theoretically valuable and computationally efficient [32]. The exact solutions are also valuable to researchers and engineers as they can serve as benchmark results for checking the validation and accuracy of numerical solutions. Moreover, the exact solutions can show clearly the intrinsic features of the solutions with respect to the various design parameters [33]. Thus the present study is necessary and important.

The organization of this paper is as follows. In Section 2 the equilibrium equations and boundary conditions are presented. In Section 3, the separation of variables method and the eigenfunctions for simple support plates and arbitrary opposite edges $y = 0$ and $b$ are presented. Then numerical comparisons with other solutions are presented in Section 4. Finally, conclusions are outlined in Section 5.

## 2. Equilibrium equations and boundary conditions

Consider a thick rectangular plate of length $a$, width $b$ and uniform thickness $h$, oriented so that its undeformed middle surface contains the $x$ and $y$ axes of a Cartesian coordinate system $(x,y,z)$, as shown in Fig. 1. Three fundamental variables in classical Mindlin plate theory (MPT) are the displacements along $x$, $y$ and $z$ directions, as

$$u = -z\psi_x(x,y,z,t), \quad v = -z\psi_y(x,y,z,t), \quad w = w(x,y,z,t) \tag{1}$$

where $t$ is the time coordinate, $w$ the deflection, and $\psi_x$ and $\psi_y$ are the angles of rotations of a normal line due to plate bending with respect to $y$ and $x$ coordinates, respectively. The relations between the internal forces and displacements for orthotropic Mindlin plates are

$$M_x = -\left(D_{11}\frac{\partial\psi_x}{\partial x} + D_{12}\frac{\partial\psi_y}{\partial y}\right), \quad M_y = -\left(D_{22}\frac{\partial\psi_y}{\partial y} + D_{21}\frac{\partial\psi_x}{\partial x}\right) \tag{2a}$$

$$M_{xy} = -D_{66}\left(\frac{\partial\psi_x}{\partial y} + \frac{\partial\psi_y}{\partial x}\right) \tag{2b}$$

$$Q_y = C_{44}\left(\frac{\partial w}{\partial y} - \psi_y\right), \quad Q_x = C_{55}\left(\frac{\partial w}{\partial x} - \psi_x\right) \tag{2c}$$

where

$$\begin{aligned} D_{11} &= \frac{E_x h^3}{12(1-\upsilon_x\upsilon_y)}, \quad D_{12} = \frac{\upsilon_x E_x h^3}{12(1-\upsilon_x\upsilon_y)}, \\ D_{22} &= \frac{E_y h^3}{12(1-\upsilon_x\upsilon_y)}, \quad D_{21} = \frac{\upsilon_y E_y h^3}{12(1-\upsilon_x\upsilon_y)} \\ D_{66} &= \frac{G_{xy}h^3}{12}, \quad C_{44} = \kappa G_{yz}h, \quad C_{55} = \kappa G_{zx}h \end{aligned} \tag{3}$$

are the bending and shear rigidities, respectively, and $\kappa$ the shear correction factor. In view of the Betti Principle the product $\upsilon_x E_x = \upsilon_y E_y$, therefore $D_{12} = D_{21}$. The above bending and shear rigidities are given for one layer. The formulations of rigidities of laminates can be found in text books or research papers, for example, [4]. The equations of free motion in absence of the external loads are given by

$$-\frac{\partial M_x}{\partial x} - \frac{\partial M_{xy}}{\partial y} + Q_x - \rho J\frac{\partial^2\psi_x}{\partial t^2} = 0 \tag{4a}$$

$$-\frac{\partial M_{xy}}{\partial x} - \frac{\partial M_y}{\partial y} + Q_y - \rho J\frac{\partial^2\psi_y}{\partial t^2} = 0 \tag{4b}$$

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} - \rho h\frac{\partial^2 w}{\partial t^2} = 0 \tag{4c}$$

where $J = h^3/12$ is area axial moment of inertia of cross section per unit length, $\rho$ the volume density. For principle or harmonic vibration, it is assumed that

$$\psi_x = \psi_x(x,y)e^{i\omega t}, \quad \psi_y = \psi_y(x,y)e^{i\omega t}, \quad w = w(x,y)e^{i\omega t} \tag{5}$$

Substitution of expressions Eq. (5) into Eq. (2) and then Eq. (2) into Eq. (4) leads to the eigenvalue partial differential equations in terms of displacements, as

$$\left.\begin{aligned} &D_1\frac{\partial^2\psi_x}{\partial x^2} + \frac{\partial^2\psi_x}{\partial y^2} + D_3\frac{\partial^2\psi_y}{\partial x\partial y} + C_1\left(\frac{\partial w}{\partial x} - \psi_x\right) + \gamma^4\psi_x = 0 \\ &D_2\frac{\partial^2\psi_y}{\partial y^2} + \frac{\partial^2\psi_y}{\partial x^2} + D_3\frac{\partial^2\psi_x}{\partial x\partial y} + C_2\left(\frac{\partial w}{\partial y} - \psi_y\right) + \gamma^4\psi_y = 0 \\ &C_1\left(\frac{\partial^2 w}{\partial x^2} - \frac{\partial\psi_x}{\partial x}\right) + C_2\left(\frac{\partial^2 w}{\partial y^2} - \frac{\partial\psi_y}{\partial y}\right) + \beta^4 w = 0 \end{aligned}\right\} \tag{6a, b, c}$$

where

$$D_1 = \frac{D_{11}}{D_{66}}, \quad D_2 = \frac{D_{22}}{D_{66}}, \quad D_3 = \frac{D_{12}+D_{66}}{D_{66}}, \quad C_1 = \frac{C_{55}}{D_{66}}, \quad C_2 = \frac{C_{44}}{D_{66}} \tag{7a}$$

$$\gamma^4 = \frac{\rho J\omega^2}{D_{66}}, \quad \beta^4 = \frac{\rho h\omega^2}{D_{66}} \tag{7b}$$

are normalized material parameters and frequency parameters, respectively.

The three governing differential equations are of the sixth order in total so there are three boundary conditions on each edge. The boundary conditions which have to be imposed on the problem include clamped edge, free edge and simple support edge:

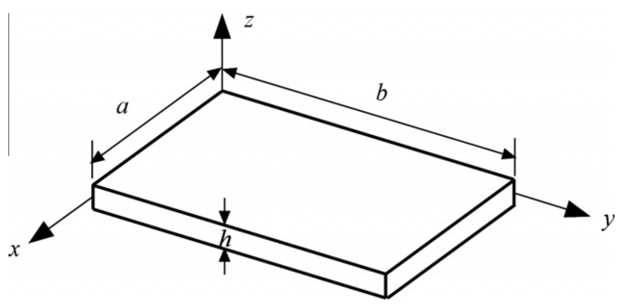


**Fig. 1.** Mindlin plate and coordinates.

(1) Simply supported edge (S), for which, $M_n = 0$, $\psi_s = 0$ and $w = 0$. They can be further simplified as

$$w = 0, \quad \psi_s = 0$$
$$M_n = 0 \Rightarrow \frac{\partial \psi_n}{\partial n} + \upsilon_n \frac{\partial \psi_s}{\partial s} = 0 \Rightarrow \frac{\partial \psi_n}{\partial n} = 0. \tag{8}$$

(2) Clamped edge (C), where displacements on the boundary are given zero values, that is, the three displacements

$$w = 0, \quad \psi_s = 0, \quad \psi_n = 0. \tag{9}$$

(3) Free edge (F), where stress resultants $Q_n$, $M_n$ and $M_{ns}$ are assigned zeros values. They can be written as

$$\frac{\partial \psi_n}{\partial n} + \upsilon_n \frac{\partial \psi_s}{\partial s} = 0, \quad \frac{\partial \psi_n}{\partial s} + \frac{\partial \psi_s}{\partial n} = 0, \quad \frac{\partial w}{\partial n} - \psi_n = 0. \tag{10}$$

This paper is dedicated to find exact solutions of Eqs. (6) with one pair of opposite edges being simply supported, while the other pair of opposite edges being any combinations of the three types of boundary conditions.

## 3. Exact solutions

This section presents the separation of variables method for exactly solving free vibrations of rectangular orthotropic Mindlin plates. The method was proposed by Xing and Liu [3,30] to solve the exact solutions for free lateral vibrations of rectangular thin orthotropic and isotropic plates with any combinations of simply supported and clamped edges. The method is again employed here to solve the titled problem. The exact solutions for free vibrations of isotropic Mindlin plates are also discussed due to their close relations with those of orthotropic Mindlin plates and some distinct characteristics compared with the exact solutions of orthotropic Mindlin plates.

To eliminate $\psi_y$ from Eq. (6a) by substitution of Eq. (6c) into Eq. (6a) yields

$$\left[1 - \frac{1}{C_1}\left(D_1 \frac{\partial^2}{\partial x^2} - D_3 \frac{C_1}{C_2} \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \gamma^4\right)\right]\psi_x = \left[1 + \frac{D_3}{C_1}\left(\frac{\partial^2}{\partial y^2} + \frac{C_1}{C_2}\frac{\partial^2}{\partial x^2} + \frac{\beta^4}{C_2}\right)\right]\frac{\partial w}{\partial x} \tag{11a}$$

Similarly, we can obtain

$$\left[1 - \frac{1}{C_2}\left(D_2 \frac{\partial^2}{\partial y^2} - D_3 \frac{C_2}{C_1} \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial x^2} + \gamma^4\right)\right]\psi_y = \left[1 + \frac{D_3}{C_2}\left(\frac{\partial^2}{\partial x^2} + \frac{C_2}{C_1}\frac{\partial^2 w}{\partial y^2} + \frac{\beta^4}{C_1}\right)\right]\frac{\partial w}{\partial y} \tag{11b}$$

The differentiation of Eq. (6c) with respect to both $x$ and $y$ results in

$$\left(C_1 \frac{\partial^2}{\partial x^2} + C_2 \frac{\partial^2}{\partial y^2} + \beta^4\right)\frac{\partial w}{\partial x \partial y} = C_1 \frac{\partial^2}{\partial x^2}\frac{\partial \psi_x}{\partial y} + C_2 \frac{\partial^2}{\partial y^2}\frac{\partial \psi_y}{\partial x} \tag{11c}$$

Eqs. (11) can also be written as

$$\left.\begin{array}{l} L_1 \frac{\partial \psi_x}{\partial y} = K_1 \frac{\partial w}{\partial x \partial y}, \quad L_2 \frac{\partial \psi_y}{\partial x} = K_2 \frac{\partial w}{\partial x \partial y} \\ K_3 \frac{\partial w}{\partial x \partial y} = C_1 \frac{\partial^2}{\partial x^2}\frac{\partial \psi_x}{\partial y} + C_2 \frac{\partial^2}{\partial y^2}\frac{\partial \psi_y}{\partial x} \end{array}\right\} \tag{12a, b, c}$$

where

$$L_1 = 1 - \frac{1}{C_1}\left(D_1 \frac{\partial^2}{\partial x^2} - D_3 \frac{C_1}{C_2}\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \gamma^4\right) \tag{13a}$$

$$L_2 = 1 - \frac{1}{C_2}\left(D_2 \frac{\partial^2}{\partial y^2} - D_3 \frac{C_2}{C_1}\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial x^2} + \gamma^4\right) \tag{13b}$$

$$K_1 = 1 + \frac{D_3}{C_1}\left(\frac{\partial^2}{\partial y^2} + \frac{C_1}{C_2}\frac{\partial^2}{\partial x^2} + \frac{\beta^4}{C_2}\right) \tag{13c}$$

$$K_2 = 1 + \frac{D_3}{C_2}\left(\frac{\partial^2}{\partial x^2} + \frac{C_2}{C_1}\frac{\partial^2 w}{\partial y^2} + \frac{\beta^4}{C_1}\right) \tag{13d}$$

$$K_3 = C_1 \frac{\partial^2}{\partial x^2} + C_2 \frac{\partial^2}{\partial y^2} + \beta^4 \tag{13e}$$

To eliminate both $\psi_x$ and $\psi_y$ from Eqs. ((12a)–(c)) yields

$$\left(L_1 L_2 K_3 - C_1 K_1 L_2 \frac{\partial^2}{\partial x^2} - C_2 L_1 K_2 \frac{\partial^2}{\partial y^2}\right)\frac{\partial w}{\partial x \partial y} = 0 \tag{14a}$$

Similarly, we can obtain

$$\left(L_1 L_2 K_3 - C_1 K_1 L_2 \frac{\partial^2}{\partial x^2} - C_2 L_1 K_2 \frac{\partial^2}{\partial y^2}\right)\frac{\partial \psi_x}{\partial y} = 0 \tag{14b}$$

$$\left(L_1 L_2 K_3 - C_1 K_1 L_2 \frac{\partial^2}{\partial x^2} - C_2 L_1 K_2 \frac{\partial^2}{\partial y^2}\right)\frac{\partial \psi_y}{\partial x} = 0 \tag{14c}$$

Assume the pair of opposite edges of $x = 0$ and $a$ as simply supported. To substitute the separation of variables solution $w = e^{\mu x} e^{\lambda y}$, where $\mu = i\alpha$, $i = \sqrt{-1}$, $\alpha = m\pi/a$, $m$ is the number of half-waves in the $x$ direction, into the Eq. (14a) yields

$$\bar{a}\lambda^6 + \bar{b}\lambda^4 + \bar{c}\lambda^2 + \bar{d} = 0 \tag{15}$$

where

$$\bar{a} = \frac{D_2}{C_1} \tag{16a}$$

$$\bar{b} = \frac{1}{C_1 C_2}\left[C_2(1 + D_2)\gamma^4 + (\beta^4 - C_1 C_2)D_2\right] - \frac{1}{C_1 C_2}\left[C_2\left(D_1 D_2 + 1 - D_3^2\right) + C_1 D_2\right]\alpha^2 \tag{16b}$$

$$\begin{aligned}\bar{c} = {} & \left[C_2(\gamma^8 - g) - h + (D_2 + 1)\beta^4\gamma^4 - C_1 D_2 \beta^4\right]\frac{1}{C_1 C_2} \\ & + \left[C_2 D_1 + C_1\left(1 + D_1 D_2 - D_3^2\right)\right]\frac{\alpha^4}{C_1 C_2} \\ & - \left\{\left[C_2(1 + D_1) + C_1(1 + D_2)\right]\gamma^4 + \left(1 - D_3^2 + D_1 D_2\right)\beta^4 \right. \\ & \left. - 2C_1 C_2(D_3 + 1)\right\}\frac{\alpha^2}{C_1 C_2}\end{aligned} \tag{16c}$$

$$\begin{aligned}\bar{d} = {} & \left[C_1 C_2 + (\gamma^4 - C_2 - C_1)\gamma^4\right]\frac{\beta^4}{C_1 C_2} \\ & + \left[C_1(1 + D_1)\gamma^4 \right. \\ & \left. + D_1(\beta^4 - C_1 C_2)\right]\frac{\alpha^4}{C_1 C_2} - \left[C_1(\gamma^8 - \beta^4) + (1 + D_1)\beta^4\gamma^4 \right. \\ & \left. - C_2(C_1\gamma^4 + D_1\beta^4)\right]\frac{\alpha^2}{C_1 C_2} - \frac{D_1 \alpha^6}{C_2}\end{aligned} \tag{16d}$$

Let

$$\lambda^2 = s - \frac{\bar{b}}{3\bar{a}}. \tag{17}$$

The substitution of Eq. (17) into Eq. (15) yields

$$s^3 + ps + q = 0 \tag{18}$$

where

$$p = \frac{1}{\bar{a}}\left(\bar{c} - \frac{\bar{b}^2}{3\bar{a}}\right), \quad q = \frac{1}{\bar{a}}\left(\bar{d} + \frac{2\bar{b}^3}{27\bar{a}^2} - \frac{\bar{b}\bar{c}}{3\bar{a}}\right) \tag{19}$$

The roots of Eq. (18) are

$$s_1 = \Delta_1 + \Delta_2, \quad s_2 = \varpi\Delta_1 + \varpi^2\Delta_2, \quad s_3 = \varpi^2\Delta_1 + \varpi\Delta_2 \tag{20a, b, c}$$

where

$$\Delta_1 = \sqrt[3]{-\frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{2}\right)^3}}, \quad \Delta_2 = \sqrt[3]{-\frac{q}{2} - \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{2}\right)^3}} \tag{21}$$

$$\varpi = \frac{-1 + \mathrm{i}\sqrt{3}}{2}, \quad \varpi^2 = \frac{-1 - \mathrm{i}\sqrt{3}}{2} \tag{22}$$

So the roots of Eq. (15) can be expressed as

$$\lambda_{1,2} = \pm \mathrm{i}\beta_1, \quad \lambda_{3,4} = \pm \mathrm{i}\beta_2, \quad \lambda_{5,6} = \pm \mathrm{i}\beta_3 \tag{23}$$

where

$$\beta_j = \sqrt{\frac{\bar{b}}{3\bar{a}} - s_j} \quad j = 1, 2, 3 \tag{24}$$

Whether $\beta_j$, $j$ = 1, 2, 3 can be real, imaginary or complex which cannot be directly determined through the above derivations. We assume they are real or imaginary but not complex, of which the correctness can be verified through calculation. Then the eigenfunctions $w$, $\psi_x$ and $\psi_y$ in separation of variable form can be expressed in terms of the eigenvalues by three potentials $W_j$, $j$ = 1, 2, 3, as

$$\psi_x = \sum_{j=1}^{3} g_j \frac{\partial W_j}{\partial x}, \quad \psi_y = \sum_{j=1}^{3} h_j \frac{\partial W_j}{\partial y}, \quad w = \sum_{j=1}^{3} W_j \tag{25a, b, c}$$

where

$$W_j(x, y) = (A_j \sin\beta_j y + B_j \cos\beta_j y)\sin\alpha x \tag{26}$$

The substitution of Eqs. ((25a)–(c)) into Eqs. (11a,b) leads to

$$g_j = \left[1 - \frac{D_3}{C_1}\left(\beta_j^2 + \frac{C_1}{C_2}\alpha^2 - \frac{\beta^4}{C_2}\right)\right] \times \left[1 + \frac{1}{C_1}\left(D_1\alpha^2 - D_3\frac{C_1}{C_2}\alpha^2 + \beta_j^2 - \gamma^4\right)\right]^{-1} \tag{27a}$$

$$h_j = \left[1 - \frac{D_3}{C_2}\left(\alpha^2 + \frac{C_2}{C_1}\beta_j^2 - \frac{\beta^4}{C_1}\right)\right] \times \left[1 + \frac{1}{C_2}\left(D_2\beta_j^2 - D_3\frac{C_2}{C_1}\beta_j^2 + \alpha^2 - \gamma^4\right)\right]^{-1} \tag{27b}$$

where $j$ = 1, 2, 3. To substitute Eqs. (25)–(27) into Eq. (11c) leads to

$$\bar{a}\beta_j^6 - \bar{b}\beta_j^4 + \bar{c}\beta_j^2 - \bar{d} = 0 \quad j = 1, 2, 3 \tag{28}$$

Since we solved $\beta_j$, $j$ = 1, 2, 3 from Eq. (15), the correctness of Eqs. (28) is straight forward, in other words, the exactness of Eqs. ((25a)–(c)) is proved.

The governing differential equations of the deflection and the two rotations of isotropic plates are distinct from those of orthotropic plates. They can be derived as

$$\left[\frac{\upsilon_1 D}{C}\nabla^2 - \left(1 - \frac{\omega^2\rho J}{C}\right)\right] \times \left[\nabla^2\nabla^2 + \left(\frac{J}{h} + \frac{D}{C}\right)\frac{\omega^2\rho h}{D}\nabla^2 - \left(1 - \frac{\omega^2\rho J}{C}\right)\frac{\omega^2\rho h}{D}\right]\frac{\partial\psi_x}{\partial y} = 0, \tag{29a}$$

$$\left[\frac{\upsilon_1 D}{C}\nabla^2 - \left(1 - \frac{\omega^2\rho J}{C}\right)\right] \times \left[\nabla^2\nabla^2 + \left(\frac{J}{h} + \frac{D}{C}\right)\frac{\omega^2\rho h}{D}\nabla^2 - \left(1 - \frac{\omega^2\rho J}{C}\right)\frac{\omega^2\rho h}{D}\right]\frac{\partial\psi_y}{\partial x} = 0, \tag{29b}$$

$$\left[\nabla^2\nabla^2 + \left(\frac{D}{C} + \frac{J}{h}\right)\frac{\omega^2\rho h}{D}\nabla^2 - \left(1 - \frac{\rho J\omega^2}{C}\right)\frac{\omega^2\rho h}{D}\right]w = 0, \tag{29c}$$

where $\upsilon_1 = (1 - \upsilon)/2$, $\upsilon_2 = (1 + \upsilon)/2$, $\upsilon$ the Poisson's ratio, $D = Eh^3/12(1 - \upsilon^2)$ the flexural rigidity, $C = \kappa Gh$ the shear rigidity. We can see that the differential operator of Eq. (29c) is a biquadratic factor of Eqs. (29a,b). Thus the solutions for the lateral deflection $w$ and the two rotations are of different order, which is the particularity of isotropic Mindlin plates. One can find in the work of Mindlin et al. [9,16], and Shahrokh and Arsanjani [18] that for isotropic plates, $W_3$ generates no deflection but it produces rotations.

In the following we will give the exact solutions for free vibrating orthotropic Mindlin plates. This will be straightforward for simply supported plate because its eigenvalues are known and the eigenfunctions can be simply expressed as

$$w = W_1, \quad \psi_x = g_1\frac{\partial W_1}{\partial x}, \quad \psi_y = h_1\frac{\partial W_1}{\partial y} \tag{30a, b, c}$$

The substitution of Eqs. ((30a)–(c)) into Eq. (14a) results in a cubic algebraic equation with respect to $\xi = \beta^4$ which can be solved using the same procedure as Eqs. (15)–(24). The three roots are

$$\xi_j = \sqrt{\tilde{s}_j - \frac{\tilde{b}}{3\tilde{a}}} \quad j = 1, 2, 3 \tag{31}$$

where $\xi_j$, $j$ = 1, 2, 3 are different frequency parameters of simply supported plates, $\tilde{s}_j$, $j$ = 1, 2, 3 are similar as Eqs. (18)–(22) but

$$\tilde{a} = \frac{\bar{J}^2}{C_1C_2} \tag{32a}$$

$$\tilde{b} = -\frac{\bar{J}}{C_1C_2}\left[C_1 + C_2 + (1 + D_1 + C_1\bar{J})\alpha^2 + (1 + D_2 + C_2\bar{J})\beta_1^2\right] \tag{32b}$$

$$\tilde{c} = \frac{1}{C_1C_2}\{C_1C_2 + [C_1 + (C_1\bar{J} + D_1)C_2]\alpha^2 + [C_2 + C_1(C_2\bar{J} + D_2)]\beta_1^2 + [C_2\bar{J}(1 + D_1) + D_1D_2 - D_3^2 + C_1\bar{J}(1 + D_2)]\alpha^2\beta_1^2 + [D_1 + C_1\bar{J}(1 + D_1)]\alpha^4 + [D_2 + C_2\bar{J}(1 + D_2)]\beta_1^4\} \tag{32c}$$

$$\tilde{d} = -\frac{1}{C_1C_2}\{C_1D_1(\alpha^2 + C_2)\alpha^4 + C_2D_2(\beta_1^2 + C_1)\beta_1^4 + 2C_1C_2(1 + D_3)\alpha^2\beta_1^2 + \left[C_1\left(1 - D_3^2 + D_1D_2\right) + C_2D_1\right]\alpha^4\beta_1^2 + \left[C_2\left(1 - D_3^2 + D_1D_2\right) + C_1D_2\right]\alpha^2\beta_1^4\} \tag{32d}$$

where $\bar{J} = h^2/12$, $\alpha = m\pi/a$, $\beta_1 = n\pi/b$, $n$ is the number of half-waves in the $y$ direction.

For plates with $x$ = 0, $a$ simply supported while $y$ = 0, $b$ arbitrary, the exact eigenequations and eigenfunctions can be obtained by substituting Eqs. (25) into boundary conditions. The eigenequations for the other five cases, namely, $y$= 0, $b$ being S–C, C–C, S–F, C–F, and F–F, are listed below: Case S–C:

$$[M_1h_3\beta_3\sin(b\beta_1) + M_3h_1\beta_1\cos(b\beta_1)\tan(b\beta_3)]\tan(b\beta_2) - M_2h_2\beta_2\tan(b\beta_3)\sin(b\beta_1) = 0. \tag{33}$$

where

$$M_1 = g_1 - g_2, \quad M_2 = g_1 - g_3, \quad M_3 = g_2 - g_3. \tag{34}$$

Case C–C:

$$2h_1h_2\beta_1\beta_2\tan(b\beta_3)M_2M_3[\cos(b\beta_1) - 1/\cos(b\beta_2)] - 2h_1h_3\beta_1\beta_3\tan(b\beta_2)M_1M_3[\cos(b\beta_1) - 1/\cos(b\beta_3)] + 2h_2h_3\beta_2\beta_3\sin(b\beta_1)M_1M_2\{1 - 1/[\cos(b\beta_2)\cos(b\beta_3)]\} + \sin(b\beta_1)\tan(b\beta_2)\tan(bs)[h_2^2\beta_2^2M_2^2 + h_3^2\beta_3^2M_1^2 + h_1^2\beta_1^2M_3^2] = 0. \tag{35}$$

Case S–F:

$$L_3\chi_1\sin(b\beta_1) - L_1\chi_2\cos(b\beta_1)\tan(b\beta_2) + L_2\chi_3\cos(b\beta_1)\tan(b\beta_3) = 0. \quad (36)$$

where

$$\begin{aligned}
&L_1 = \chi_7\chi_6 - \chi_9\chi_4, \quad L_2 = \chi_7\chi_5 - \chi_8\chi_4, \quad L_3 = \chi_8\chi_6 - \chi_9\chi_5,\\
&\chi_1 = g_1\upsilon_y\alpha^2 + h_1\beta_1^2, \quad \chi_2 = g_2\upsilon_y\alpha^2 + h_2\beta_2^2, \quad \chi_3 = g_3\upsilon_y\alpha^2 + h_3\beta_3^2,\\
&\chi_4 = (g_1+h_1)\alpha\beta_1, \quad \chi_5 = (g_2+h_2)\alpha\beta_2, \quad \chi_6 = (g_3+h_3)\alpha\beta_3,\\
&\chi_7 = (1-h_1)\beta_1, \quad \chi_8 = (1-h_2)\beta_2, \quad \chi_9 = (1-h_3)\beta_3.
\end{aligned} \quad (37)$$

Case C–F:

$$\begin{aligned}
&\cos(b\beta_1)(h_1\beta_1\chi_1M_3L_3 + h_2\beta_2\chi_2M_2L_1 + h_3\beta_3\chi_3M_1L_2)\\
&+ \sin(b\beta_1)\tan(b\beta_2)(h_1\beta_1\chi_2M_3L_1 + h_2\beta_2\chi_1M_2L_3)\\
&- \cos(b\beta_1)(h_2\beta_2\chi_3M_1L_1 + h_3\beta_3\chi_2M_2L_2)/[\cos(b\beta_2)\cos(b\beta_3)]\\
&- (h_2\beta_2\chi_1M_3L_1 + h_1\beta_1\chi_2M_2L_3)/\cos(b\beta_2)\\
&+ (h_1\beta_1\chi_3M_1L_3 + h_3\beta_3\chi_1M_3L_2)/\cos(b\beta_3)\\
&+ \cos(b\beta_1)\tan(b\beta_2)\tan(b\beta_3)(h_3\beta_3\chi_2M_1L_1 + h_2\beta_2\chi_3M_2L_2)\\
&- \sin(b\beta_1)\tan(b\beta_3)(h_3\beta_3\chi_1M_1L_3 + h_1\beta_1\chi_3M_3L_2) = 0.
\end{aligned} \quad (38)$$

Case F–F:

$$\begin{aligned}
&2\chi_1\chi_3\sin(b\beta_1)L_1L_2\{1 - 1/[\cos(b\beta_2)\cos(b\beta_3)]\}\\
&+ 2\chi_1\chi_2\tan(b\beta_3)L_1L_3[\cos(b\beta_1) - 1/\cos(b\beta_2)]\\
&- 2\chi_1\chi_3\tan(b\beta_2)L_3L_2[\cos(b\beta_1) - 1/\cos(b\beta_3)]\\
&+ \sin(b\beta_1)\tan(b\beta_2)\tan(b\beta_3)\left(\chi_2^2L_1^2 + \chi_3^2L_2^2 + \chi_1^2L_3^2\right) = 0.
\end{aligned} \quad (39)$$

The coefficients of eigenfunctions for the five cases can be expressed as

Case S–C:

$$\begin{aligned}
&A_1 = 1, \quad A_2 = -M_2\sin(b\beta_1)/[\sin(b\beta_2)M_3],\\
&A_3 = M_1\sin(b\beta_1)/[\sin(b\beta_3)M_3],\\
&B_1 = 0, \quad B_2 = 0, \quad B_3 = 0.
\end{aligned} \quad (40)$$

Case C–C:

$$\begin{aligned}
&A_3 = -\frac{P_1}{P_2}, \quad B_3 = \frac{P_3}{P_2}, \quad A_1 = 1, \quad B_1 = \frac{B_3M_3}{M_1},\\
&A_2 = -\frac{h_1\beta_1 + A_3h_3\beta_3}{h_2\beta_2}, \quad B_2 = -\frac{B_3M_2}{M_1}.
\end{aligned} \quad (41)$$

where

$$\begin{aligned}
P_1 &= -h_1\beta_1\tan(b\beta_2)M_1M_3[\cos(b\beta_1) - \cos(b\beta_3)]\\
&\quad + h_2\beta_2\sin(b\beta_1)M_1M_2[1 - \cos(b\beta_3)/\cos(b\beta_2)]\\
P_2 &= -h_3\beta_3\tan(b\beta_2)M_1M_3[\cos(b\beta_1) - \cos(b\beta_3)]\\
&\quad + h_2\beta_2\sin(b\beta_3)M_2M_3[\cos(b\beta_1)/\cos(b\beta_2) - 1]\\
P_3 &= h_3\beta_3\sin(b\beta_1)\tan(b\beta_2)M_1^2 + h_1\beta_1\tan(b\beta_2)\sin(b\beta_3)M_1M_3\\
&\quad - h_2\beta_2\sin(b\beta_1)\sin(b\beta_3)M_1M_2/\cos(b\beta_2).
\end{aligned} \quad (42)$$

Case S–F:

$$\begin{aligned}
&A_1 = 1, \quad B_1 = 0, \quad B_2 = 0, \quad B_3 = 0,\\
&A_2 = -[\chi_6\chi_1\sin(b\beta_1) - \chi_4\chi_3\cos(b\beta_1)\tan(b\beta_3)]/[\chi_6\chi_2\sin(b\beta_2)\\
&\quad - \chi_5\chi_3\cos(b\beta_2)\tan(b\beta_3)],\\
&A_3 = -[\chi_4\chi_2\cos(b\beta_1)\tan(b\beta_2) - \chi_5\chi_1\sin(b\beta_1)]/[\chi_6\chi_2\cos(b\beta_3)\\
&\quad \tan(b\beta_2) - \chi_5\chi_3\sin(b\beta_3)].
\end{aligned} \quad (43)$$

Case C–F:

$$\begin{aligned}
&A_3 = -\frac{P_4}{P_5}, \quad B_3 = -\frac{P_6}{P_5}, \quad A_1 = 1, \quad B_1 = \frac{B_3M_3}{M_1},\\
&A_2 = -\frac{h_1\beta_1 + A_3h_3\beta_3}{h_2\beta_2}, \quad B_2 = -\frac{B_3M_2}{M_1}.
\end{aligned} \quad (44)$$

where

$$\begin{aligned}
P_4 &= -h_1\beta_1[\chi_5\chi_1\cos(b\beta_1)\cos(b\beta_2) + \chi_4\chi_2\sin(b\beta_1)\sin(b\beta_2)]M_3\\
&\quad + h_2\chi_4\beta_2\chi_1M_3 - h_2\beta_2[\chi_4\chi_2\cos(b\beta_1)\cos(b\beta_2)\\
&\quad + \chi_5\chi_1\sin(b\beta_1)\sin(b\beta_2)]M_2 + h_1\chi_5q\chi_2M_2\\
&\quad + \chi_3\cos(b\beta_3)[h_2\chi_4\beta_2\cos(b\beta_1) - h_1\chi_5\beta_1\cos(b\beta_2)]M_1\\
&\quad + \chi_6\sin(b\beta_3)[h_2\beta_2\chi_1\sin(b\beta_1) - h_1\beta_1\chi_2\sin(b\beta_2)]M_1,
\end{aligned} \quad (45a)$$

$$\begin{aligned}
P_5 &= h_2\chi_6\beta_2\chi_3M_1 + h_3\chi_5\beta_3\chi_2M_2 + \chi_1\cos(b\beta_1)M_3[h_2\chi_6\beta_2\cos(b\beta_3)\\
&\quad - h_3\chi_5\beta_3\cos(b\beta_2)] - \chi_4\sin(b\beta_1)[h_3\beta_3\chi_2\sin(br)\\
&\quad - h_2\beta_2\chi_3\sin(bs)]M_3 - h_2\beta_2[\chi_6\chi_2\cos(b\beta_2)\cos(b\beta_3)\\
&\quad + \chi_5\chi_3\sin(b\beta_2)\sin(b\beta_3)]M_2 - h_3\beta_3[\chi_5\chi_3\cos(b\beta_2)\cos(b\beta_3)\\
&\quad + \chi_6\chi_2\sin(b\beta_2)\sin(b\beta_3)]M_1,
\end{aligned} \quad (45b)$$

$$\begin{aligned}
P_6 &= \chi_1\sin(b\beta_1)M_1[h_2\chi_6\beta_2\cos(b\beta_3) - h_3\chi_5\beta_3\cos(b\beta_2)]\\
&\quad - \chi_2\sin(b\beta_2)M_1[h_1\chi_6\beta_1\cos(b\beta_3) - h_3\chi_4\beta_3\cos(b\beta_1)]\\
&\quad - \chi_3\sin(b\beta_3)M_1[h_2\chi_4\beta_2\cos(b\beta_1) - h_1\chi_5\beta_1\cos(b\beta_2)].
\end{aligned} \quad (45c)$$

Case F-F:

$$\begin{aligned}
&B_1 = -\frac{P_7}{P_8}, \quad B_2 = \frac{P_9}{P_8}, \quad A_1 = 1, \quad A_2 = -\frac{L_1}{L_3}, \quad A_3 = \frac{L_2}{L_3},\\
&B_3 = -\frac{B_1\chi_1 + B_2\chi_2}{z}.
\end{aligned} \quad (46)$$

where

$$\begin{aligned}
P_7 &= \chi_1\sin(b\beta_1)[\chi_6\chi_2\sin(b\beta_3) - \chi_4\chi_3\sin(b\beta_2)]L_3\\
&\quad - \chi_5\chi_2\chi_3L_1[\cos(b\beta_2)\cos(b\beta_3) - 1] - \sin(b\beta_2)\sin(b\beta_3)\\
&\quad \times(\chi_6\chi_2^2L_1 + n\chi_3^2L_2) - \chi_6\chi_2\chi_3L_2[\cos(b\beta_2)\cos(b\beta_3) - 1]\\
&\quad - \chi_4\chi_2\chi_3\cos(b\beta_1)L_3[\cos(b\beta_2) - \cos(b\beta_3)],
\end{aligned} \quad (47a)$$

$$\begin{aligned}
P_8 &= \chi_6\chi_1\chi_2\sin(b\beta_3)L_3[\cos(b\beta_1) - \cos(b\beta_2)]\\
&\quad - \chi_5\chi_1\chi_3\sin(b\beta_2)L_3[\cos(b\beta_1) - \cos(b\beta_3)]\\
&\quad + \chi_4\chi_2\chi_3\sin(b\beta_1)L_3[\cos(b\beta_2) - \cos(b\beta_3)],
\end{aligned} \quad (47b)$$

$$\begin{aligned}
P_9 &= \sin(b\beta_1)\sin(b\beta_3)(\chi_6\chi_1^2L_3 - \chi_4\chi_3^2L_2)\\
&\quad + \chi_4\chi_1\chi_3L_3[\cos(b\beta_1)\cos(b\beta_3) - 1]\\
&\quad + \chi_5\chi_1\chi_3\cos(b\beta_2)L_1[\cos(b\beta_1) - \cos(b\beta_3)]\\
&\quad - \chi_6\chi_1\chi_3L_2[\cos(b\beta_1)\cos(b\beta_3) - 1]\\
&\quad - \chi_2L_1\sin(b\beta_2)[\chi_6\chi_1\sin(b\beta_3) - \chi_4\chi_3\sin(b\beta_1)].
\end{aligned} \quad (47c)$$

## 4. Results and discussion

This section aims at further validating the correctness of the present exact solutions through numerical comparison with available numerical results of Liew [4], and solutions calculated by the present authors using the Differential Quadrure Finite Element Method (DQFEM, [31]), although the correctness of the exact method has been proved mathematically in the former section. To avoid any comparison of the round off results which might be unrealistic, the eigenvalues are calculated in the same non-dimensional frequency parameter $\Omega$ used by Liew [4] as

**Table 1**
Frequency parameters $\Omega = (\omega b^2/\pi^2)\sqrt{\rho h/D_0}$ for three-ply laminated plates (0°,90°,0°) of SSSS square plates.

| $h/b$ | Mode sequence number | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| 0.001 | | | | | | | | |
| $\alpha a/\pi$ | 1 | 1 | 1 | 2 | 2 | 1 | 2 | 2 |
| $\beta_1 b/\pi$ | 1 | 2 | 3 | 1 | 2 | 4 | 3 | 4 |
| Exact | 6.625 | 9.447 | 16.205 | 25.115 | 26.498 | 26.657 | 30.314 | 37.785 |
| Liew | 6.625 | 9.447 | 16.205 | 25.115 | 26.498 | 26.657 | 30.314 | 37.785 |
| DQFEM | 6.625 | 9.447 | 16.205 | 25.115 | 26.498 | 26.657 | 30.314 | 37.785 |
| 0.050 | | | | | | | | |
| $\alpha a/\pi$ | 1 | 1 | 1 | 2 | 2 | 1 | 2 | 2 |
| $\beta_1 b/\pi$ | 1 | 2 | 3 | 1 | 2 | 4 | 3 | 4 |
| Exact | 6.161 | 8.899 | 15.085 | 19.572 | 20.872 | 23.970 | 24.506 | 31.094 |
| Liew | 6.138 | 8.888 | 15.110 | 19.354 | 20.665 | 24.070 | 24.344 | 31.028 |
| DQFEM | 6.161 | 8.899 | 15.085 | 19.572 | 20.872 | 23.970 | 24.506 | 31.094 |
| 0.100 | | | | | | | | |
| $\alpha a/\pi$ | 1 | 1 | 1 | 2 | 2 | 2 | 1 | 3 |
| $\beta_1 b/\pi$ | 1 | 2 | 3 | 1 | 2 | 3 | 4 | 1 |
| Exact | 5.218 | 7.773 | 12.844 | 13.313 | 14.606 | 17.915 | 19.292 | 21.562 |
| Liew | 5.166 | 7.757 | 12.915 | 13.049 | 14.376 | 17.788 | 19.502 | 21.051 |
| DQFEM | 5.218 | 7.773 | 12.844 | 13.313 | 14.606 | 17.915 | 19.292 | 21.562 |
| 0.150 | | | | | | | | |
| $\alpha a/\pi$ | 1 | 1 | 2 | 1 | 2 | 2 | 3 | 1 |
| $\beta_1 b/\pi$ | 1 | 2 | 1 | 3 | 2 | 3 | 1 | 4 |
| Exact | 4.336 | 6.672 | 9.711 | 10.721 | 10.995 | 13.854 | 15.048 | 15.431 |
| Liew | 4.275 | 6.667 | 9.488 | 10.824 | 10.826 | 13.804 | 14.665 | 15.590 |
| DQFEM | 4.336 | 6.672 | 9.711 | 10.721 | 10.995 | 13.854 | 15.048 | 15.431 |
| 0.200 | | | | | | | | |
| $\alpha a/\pi$ | 1 | 1 | 2 | 2 | 1 | 2 | 3 | 3 |
| $\beta_1 b/\pi$ | 1 | 2 | 1 | 2 | 3 | 3 | 1 | 2 |
| Exact | 3.652 | 5.759 | 7.578 | 8.809 | 9.025 | 11.224 | 11.505 | 12.363 |
| Liew | 3.594 | 5.769 | 7.397 | 8.688 | 9.145 | 11.208 | 11.223 | 12.117 |
| DQFEM | 3.652 | 5.759 | 7.578 | 8.809 | 9.025 | 11.224 | 11.505 | 12.363 |

**Table 2**
Frequency parameters $\Omega = (\omega b^2/\pi^2)\sqrt{\rho h/D_0}$ for three-ply laminated plates (0°, 90°, 0°) of SCSC square plates.

| $h/b$ | Mode sequence number | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| 0.05 | | | | | | | | |
| $\alpha a/\pi$ | 1 | 1 | 1 | 2 | 2 | 2 | 1 | 2 |
| $\beta_1 b/\pi$ | 1.425 | 2.440 | 3.424 | 1.341 | 2.400 | 3.400 | 4.396 | 4.379 |
| $\beta_2 b/\pi$ | 21.540i | 21.533i | 21.516i | 2.100i | 25.587i | 3.503i | 21.481i | 25.540i |
| $\beta_3 b/\pi$ | 1.716i | 2.532i | 3.302i | 25.593i | 2.800i | 25.571i | 3.989i | 4.148i |
| Exact | 6.907 | 11.230 | 18.578 | 19.833 | 21.984 | 26.789 | 28.058 | 34.307 |
| Liew | 6.890 | 11.246 | 18.664 | 19.619 | 21.801 | 26.689 | 28.260 | 34.348 |
| DQFEM | 6.907 | 11.230 | 18.578 | 19.833 | 21.984 | 26.789 | 28.058 | 34.307 |
| 0.10 | | | | | | | | |
| $\alpha a/\pi$ | 1 | 1 | 2 | 1 | 2 | 2 | 1 | 3 |
| $\beta_1 b/\pi$ | 1.408 | 2.368 | 1.368 | 3.306 | 2.350 | 3.295 | 4.240 | 1.345 |
| $\beta_2 b/\pi$ | 12.791i | 12.762i | 18.800i | 12.693i | 18.779i | 18.731i | 12.572i | 1.771i |
| $\beta_3 b/\pi$ | 1.548i | 2.183i | 1.684i | 2.663i | 2.274i | 2.730i | 2.989i | 25.864i |
| Exact | 5.905 | 9.412 | 13.594 | 14.712 | 15.522 | 19.267 | 20.952 | 21.735 |
| Liew | 5.871 | 9.454 | 13.340 | 14.878 | 15.340 | 19.229 | 21.231 | 21.275 |
| DQFEM | 5.905 | 9.412 | 13.594 | 14.712 | 15.522 | 19.267 | 20.952 | 21.735 |
| 0.20 | | | | | | | | |
| $\alpha a/\pi$ | 1 | 1 | 2 | 2 | 1 | 2 | 3 | 3 |
| $\beta_1 b/\pi$ | 1.344 | 2.217 | 1.325 | 2.211 | 3.128 | 3.126 | 1.305 | 2.205 |
| $\beta_2 b/\pi$ | 9.383i | 9.313i | 16.652i | 16.612i | 9.169i | 16.531i | 24.342i | 24.314i |
| $\beta_3 b/\pi$ | 1.227i | 1.546i | 1.276i | 1.584i | 1.643i | 1.678i | 1.339i | 1.635i |
| Exact | 4.165 | 6.411 | 7.828 | 9.238 | 9.476 | 11.583 | 11.664 | 12.665 |
| Liew | 4.137 | 6.474 | 7.664 | 9.159 | 9.643 | 11.377 | 11.625 | 12.448 |
| DQFEM | 4.165 | 6.411 | 7.828 | 9.238 | 9.476 | 11.583 | 11.664 | 12.665 |

$$\Omega = (\omega b^2/\pi^2)\sqrt{\rho h/D_0} \tag{48}$$

where

$$D_0 = E_y h^3/12(1 - \upsilon_x \upsilon_y) \tag{49}$$

where $h$ is the total thickness. The shear correction factor $\kappa$ is taken as $\pi^2/12$. Thick symmetric cross-ply laminates with layers of equal thickness are considered. The material properties for all layers of the laminates are identical: $E_x/E_y = 40$, $G_{23} = 1/2\ E_y$, $G_{12} = G_{31} = 3/5E_y$, $\upsilon_y = 1/4$, $\upsilon_x = 0.00625$.

**Table 3**
Frequency parameters $\Omega = (\omega b^2/\pi^2)\sqrt{\rho h/D_0}$ for three-ply laminated plates (0°, 90°, 0°) of SFSF square plates.

| $h/b$ | Mode sequence number | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| 0.05 | | | | | | | | |
| $\alpha a/\pi$ | 1 | 1 | 1 | 1 | 2 | 2 | 1 | 2 |
| $\beta_1 b/\pi$ | 0.115i | 0.732 | 1.591 | 2.543 | 0.168i | 0.807 | 3.531 | 1.669 |
| $\beta_2 b/\pi$ | 21.541i | 21.541i | 21.539i | 21.532i | 25.594i | 25.594i | 21.513i | 25.592i |
| $\beta_3 b/\pi$ | 0.992i | 1.234i | 1.845i | 2.614i | 1.642i | 1.828i | 3.381i | 2.302i |
| Exact | 5.756 | 5.929 | 7.357 | 11.864 | 19.339 | 19.479 | 19.519 | 20.244 |
| Liew | 5.734 | 5.933 | 7.397 | 11.918 | 19.124 | 19.284 | 19.602 | 20.086 |
| DQFEM | 5.757 | 5.956 | 7.419 | 11.921 | 19.341 | 19.500 | 19.559 | 20.299 |
| 0.10 | | | | | | | | |
| $\alpha a/\pi$ | 1 | 1 | 1 | 1 | 2 | 2 | 2 | 2 |
| $\beta_1 b/\pi$ | 0.105i | 0.689 | 1.574 | 2.547 | 0.132i | 0.751 | 1.615 | 2.570 |
| $\beta_2 b/\pi$ | 12.797i | 12.796i | 12.788i | 12.752i | 18.804i | 18.803i | 18.797i | 18.771i |
| $\beta_3 b/\pi$ | 0.819i | 1.061i | 1.664i | 2.286i | 1.116i | 1.330i | 1.837i | 2.393i |
| Exact | 4.834 | 4.968 | 6.324 | 10.311 | 13.118 | 13.210 | 13.898 | 16.233 |
| Liew | 4.781 | 4.935 | 6.319 | 10.345 | 12.851 | 12.959 | 13.677 | 16.070 |
| DQFEM | 4.835 | 4.989 | 6.363 | 10.337 | 13.118 | 13.222 | 13.924 | 16.257 |
| 0.20 | | | | | | | | |
| $\alpha a/\pi$ | 1 | 1 | 1 | 2 | 2 | 1 | 2 | 2 |
| $\beta_1 b/\pi$ | 0.077i | 0.609 | 1.574 | 0.075i | 0.686 | 2.570 | 1.602 | 2.587 |
| $\beta_2 b/\pi$ | 9.402i | 9.400i | 9.370i | 16.663i | 16.661i | 9.266i | 16.643i | 16.584i |
| $\beta_3 b/\pi$ | 0.556i | 0.796i | 1.335i | 0.706i | 0.940i | 1.609i | 1.397i | 1.648i |
| Exact | 3.279 | 3.365 | 4.645 | 7.385 | 7.451 | 7.552 | 8.154 | 10.117 |
| Liew | 3.213 | 3.311 | 4.619 | 7.195 | 7.272 | 7.599 | 8.004 | 10.043 |
| DQFEM | 3.280 | 3.376 | 4.655 | 7.385 | 7.454 | 7.553 | 8.153 | 10.107 |

**Table 4**
Frequency parameters $\Omega = (\omega b^2/\pi^2)\sqrt{\rho h/D_0}$ for three-ply laminated plates (0°, 90°, 0°) of SSSF square plates.

| $h/b$ | Mode sequence number | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| 0.05 | | | | | | | | |
| $\alpha a/\pi$ | 1 | 1 | 1 | 1 | 2 | 2 | 2 | 2 |
| $\beta_1 b/\pi$ | 0.418 | 1.310 | 2.275 | 3.266 | 0.427 | 1.356 | 2.305 | 3.284 |
| $\beta_2 b/\pi$ | 21.541i | 21.540i | 21.535i | 21.520i | 25.594i | 25.593i | 25.588i | 25.574i |
| $\beta_3 b/\pi$ | 1.081i | 1.628i | 2.398i | 3.182i | 1.702i | 2.109i | 2.734i | 3.423i |
| Exact | 5.801 | 6.650 | 10.278 | 17.223 | 19.377 | 19.848 | 21.681 | 26.076 |
| DQFEM | 5.808 | 6.679 | 10.308 | 17.246 | 19.382 | 19.870 | 21.719 | 26.115 |
| 0.10 | | | | | | | | |
| $\alpha a/\pi$ | 1 | 1 | 1 | 2 | 2 | 1 | 2 | 2 |
| $\beta_1 b/\pi$ | 0.403 | 1.296 | 2.275 | 0.419 | 1.322 | 3.274 | 2.288 | 3.282 |
| $\beta_2 b/\pi$ | 12.797i | 12.792i | 12.766i | 18.804i | 18.800i | 12.696i | 18.781i | 18.732i |
| $\beta_3 b/\pi$ | 0.914i | 1.468i | 2.126i | 1.193i | 1.656i | 2.649i | 2.240i | 2.725i |
| Exact | 4.870 | 5.669 | 8.965 | 13.143 | 13.549 | 14.516 | 15.341 | 19.208 |
| DQFEM | 4.875 | 5.689 | 8.981 | 13.145 | 13.561 | 14.524 | 15.354 | 19.216 |
| 0.20 | | | | | | | | |
| $\alpha a/\pi$ | 1 | 1 | 1 | 2 | 2 | 2 | 1 | 3 |
| $\beta_1 b/\pi$ | 0.375 | 1.285 | 2.285 | 0.410 | 1.304 | 2.294 | 3.291 | 0.439 |
| $\beta_2 b/\pi$ | 9.402i | 9.385i | 9.305i | 16.662i | 1.266i | 16.606i | 9.135i | 24.349i |
| $\beta_3 b/\pi$ | 0.665i | 1.196i | 1.561i | 0.805i | 16.652i | 1.601i | 1.639i | 0.944i |
| Exact | 3.303 | 4.058 | 6.624 | 7.403 | 7.808 | 9.419 | 10.056 | 11.381 |
| DQFEM | 3.306 | 4.065 | 6.626 | 7.403 | 7.808 | 9.416 | 10.055 | 11.380 |

The computation is carried out for three-ply laminates with stacking sequence (0°,90°,0°) for frequency parameters in all tables. In Table 1, exact solutions for the simply supported Mindlin square plates, with relative thickness ratio $h/b$ = 0.001, 0.05, 0.10, 0.15, and 0.2, are directly calculated from Eq. (31) and compared with both the $p$-Ritz solutions by Liew [4] and the DQFEM solutions. The non-dimensional eigenvalues $\alpha a/\pi$ and $\beta_1 b/\pi$ are also included in Table 1 which can be used to draw mode shapes. We can see that all digits used for comparison of the three sets solutions are the same for plates with thickness ratio $h/b$ = 0.001. For other thickness ratios of the simply supported plates, the present exact solutions are slightly larger than those of Liew [4] for most frequency parameters, while the present exact solutions are exactly the same as those of DQFEM for the simply supported plates with other thickness ratios.

In Tables 2 and 3, exact solutions for SCSC and SFSF square plates with thickness ratio $h/b$ = 0.05, 0.10, and 0.2 are compared with the $p$-Ritz solutions by Liew [4] and the DQFEM solutions calculated by the authors. We can see that the agreement among the three type solutions is quite well. In Tables 4–6 exact solutions for SSSF, SSSC and SCSF plates are compared with the DQFEM solutions since no other numerical solutions are available. Good agreement between the exact solutions and the DQFEM solutions is evident.

The eigenvalues $\alpha a/\pi$ and $\beta_j b/\pi$, $j$ = 1, 2, 3 are also included in Tables 2–6. Our assumption that $\beta_j$ cannot be complex or can be either real or imaginary is verified. It may be seen that $\beta_2$ and $\beta_3$

**Table 5**
Frequency parameters $\Omega = (\omega b^2/\pi^2)\sqrt{\rho h/D_0}$ for three-ply laminated plates (0°, 90°, 0°) of SSSC square plates.

| $h/b$ | Mode sequence number | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| 0.05 | | | | | | | | |
| $\alpha a/\pi$ | 1 | 1 | 1 | 2 | 2 | 2 | 1 | 2 |
| $\beta_1 b/\pi$ | 1.203 | 2.221 | 3.215 | 1.158 | 2.198 | 3.201 | 4.201 | 4.192 |
| $\beta_2 b/\pi$ | 21.540i | 21.536i | 21.521i | 25.593i | 25.589i | 25.575i | 21.489i | 25.547i |
| $\beta_3 b/\pi$ | 1.548i | 2.354i | 3.143i | 1.997i | 2.659i | 3.365i | 3.858i | 4.030i |
| Exact | 6.450 | 9.983 | 16.796 | 19.675 | 21.369 | 25.594 | 26.012 | 32.676 |
| DQFEM | 6.450 | 9.983 | 16.796 | 19.675 | 21.369 | 25.594 | 26.012 | 32.676 |
| 0.10 | | | | | | | | |
| $\alpha a/\pi$ | 1 | 1 | 2 | 1 | 2 | 2 | 1 | 3 |
| $\beta_1 b/\pi$ | 1.200 | 2.190 | 1.177 | 3.158 | 2.179 | 3.152 | 4.124 | 1.165 |
| $\beta_2 b/\pi$ | 12.793i | 2.074i | 18.801i | 12.707i | 2.178i | 18.741i | 12.590i | 1.670i |
| $\beta_3 b/\pi$ | 1.400i | 12.770i | 1.567i | 2.598i | 18.784i | 2.671i | 2.956i | 25.865i |
| Exact | 5.496 | 8.577 | 13.426 | 13.796 | 15.041 | 18.593 | 20.145 | 21.632 |
| DQFEM | 5.496 | 8.577 | 13.426 | 13.796 | 15.041 | 18.593 | 20.145 | 21.632 |
| 0.20 | | | | | | | | |
| $\alpha a/\pi$ | 1 | 1 | 2 | 2 | 1 | 2 | 3 | 3 |
| $\beta_1 b/\pi$ | 1.175 | 2.116 | 1.164 | 2.113 | 3.066 | 3.065 | 1.153 | 2.109 |
| $\beta_2 b/\pi$ | 9.390i | 9.324i | 16.655i | 16.618i | 9.181i | 16.538i | 24.344i | 24.318i |
| $\beta_3 b/\pi$ | 1.137i | 1.521i | 1.197i | 1.561i | 1.643i | 1.678i | 1.270i | 1.613i |
| Exact | 3.880 | 6.103 | 7.687 | 9.031 | 9.256 | 11.408 | 11.574 | 12.518 |
| DQFEM | 3.880 | 6.103 | 7.687 | 9.031 | 9.256 | 11.408 | 11.574 | 12.518 |

**Table 6**
Frequency parameters $\Omega = (\omega b^2/\pi^2)\sqrt{\rho h/D_0}$ for three-ply laminated plates (0°, 90°, 0°) of SCSF square plates.

| $h/b$ | Mode sequence number | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| 0.05 | | | | | | | | |
| $\alpha a/\pi$ | 1 | 1 | 1 | 1 | 2 | 2 | 2 | 2 |
| $\beta_1 b/\pi$ | 0.552 | 1.509 | 2.492 | 3.477 | 0.515 | 1.522 | 2.499 | 3.480 |
| $\beta_2 b/\pi$ | 21.541i | 21.539i | 21.533i | 21.515i | 25.594i | 25.592i | 25.586i | 25.569i |
| $\beta_3 b/\pi$ | 1.139i | 1.781i | 2.574i | 3.341i | 1.725i | 2.209i | 2.871i | 3.559i |
| Exact | 5.842 | 7.124 | 11.548 | 19.041 | 19.393 | 20.037 | 22.329 | 27.309 |
| DQFEM | 5.853 | 7.157 | 11.578 | 19.061 | 19.401 | 20.067 | 22.371 | 27.349 |
| 0.10 | | | | | | | | |
| $\alpha a/\pi$ | 1 | 1 | 1 | 2 | 2 | 1 | 2 | 2 |
| $\beta_1 b/\pi$ | 0.554 | 1.486 | 2.456 | 0.535 | 1.494 | 3.423 | 2.459 | 3.425 |
| $\beta_2 b/\pi$ | 0.985i | 12.790i | 12.757i | 18.804i | 18.798i | 12.681i | 18.775i | 18.722i |
| $\beta_3 b/\pi$ | 12.797i | 1.602i | 2.234i | 1.234i | 1.762i | 2.712i | 2.334i | 2.782i |
| Exact | 4.910 | 6.091 | 9.847 | 13.160 | 13.736 | 15.458 | 15.862 | 19.912 |
| DQFEM | 4.919 | 6.111 | 9.861 | 13.165 | 13.752 | 15.466 | 15.876 | 19.921 |
| 0.20 | | | | | | | | |
| $\alpha a/\pi$ | 1 | 1 | 1 | 2 | 2 | 2 | 1 | 3 |
| $\beta_1 b/\pi$ | 0.563 | 1.433 | 2.389 | 0.563 | 1.443 | 2.394 | 3.344 | 0.567 |
| $\beta_2 b/\pi$ | 9.401i | 9.379i | 9.292i | 0.876i | 1.330i | 16.599i | 9.123i | 0.995i |
| $\beta_3 b/\pi$ | 0.7686i | 1.271i | 1.581i | 16.662i | 16.648i | 1.620i | 1.636i | 24.349i |
| Exact | 3.348 | 4.339 | 6.956 | 7.424 | 7.954 | 9.649 | 10.246 | 11.395 |
| DQFEM | 3.353 | 4.344 | 6.958 | 7.426 | 7.955 | 9.646 | 10.245 | 11.397 |

are imaginary in general, while $\beta_1$ is usually real but might become imaginary as can be seen from Table 3. Another point regarding to the eigenvalues needs to be noted is that, in Tables 2–6, two of the eigenvalues are of the same order of magnitude while one is much larger than the other two, especially when the plate is very thin. But for the cases S–C and S–F, the largest eigenvalue cannot induce any computational problem. For the cases C–C, C–F, and F–F, although some mode shapes cannot be drawn due to the computational problems caused by the largest eigenvalue, the eigenvalues or the frequencies are correct if the plate relative thickness is larger than 0.001. Moreover, high accuracy closed-form solutions that are similar with those via Kirchhoff thin plate have been obtained for vibrations of orthotropic Mindlin plates, which can resolve the computational problems caused by the largest eigenvalue and greatly simplify calculations. But they are not presented here since a single paper cannot include so much content.

## 5. Conclusion

This work employed the Mindlin plate theory to investigate the free vibrations of thick orthotropic rectangular plates. Exact eigenequations were derived for the six cases having two opposite edges simply supported, including S–S–S–S, S–C–S–S, S–C–S–C, S–S–S–F, S–F–S–F and S–C–S–F plates. The deflections were also given in closed-form for all six cases, by which one can draw the mode shapes. The exact solutions were validated through both mathematical proof and numerical comparisons with other solutions. The distinct differences of governing differential equations and eigenfunctions between orthotropic and isotropic rectangular Mindlin plates were discussed as well.

## Acknowledgement

The authors gratefully acknowledge the support from the China Postdoctoral Science Foundation (Grant No. 20100470179).

## References

[1] Biancolini ME, Brutti C, Reccia L. Approximate solution for free vibrations of thin orthotropic rectangular plates. J Sound Vib 2005;288:321–44.
[2] Demasi L. Quasi-3D analysis of free vibration of anisotropic plates. Compos Struct 2006;74:449–57.
[3] Xing YF, Liu B. New exact solutions for free vibrations of thin orthotropic rectangular plates. Compos Struct 2009;89:567–74.
[4] Liew KM. Solving the vibrations of thick symmetric laminates by Reissner/Mindlin plate theory and the *p*-Ritz method. J Sound Vib 1996;198(3):343–60.
[5] Chaudhuri RA, Balaraman K, Kunukkasseril VX. A combined theoretical and experimental investigation on free vibration of thin symmetrically laminated anisotropic plates. Compos Struct 2005;67:85–97.
[6] Sakata T, Takahashi K, Bhat RB. Natural frequencies of orthotropic rectangular plates obtained by iterative reduction of the partial differential equation. J Sound Vib 1996;189:89–101.
[7] Numayr KS, Haddad RH, Haddad MA. Free vibration of composite plates using the finite difference method. Thin-Wall Struct 2004;42:399–414.
[8] Reissner E. The effect of transverse shear deformation on the bending of elastic plate. ASME J Appl Mech 1945;12:69–77.
[9] Mindlin RD. Influence rotatory inertia and shear in flexural motion of isotropic, elastic plates. ASME J Appl Mech 1951;18:31–8.
[10] Medwadowski SJ. A refined theory of elastic, orthotropic plates. ASME J Appl Mech 1958;25:437–41.
[11] Yang PC, Norris CH, Stavsky Y. Elastic wave propagation in heterogeneous plates. Int J Solids Struct 1966;2:665–84.
[12] Whitney JM, Pagano NJ. Shear deformation in heterogeneous anisotropic plates. ASME J Appl Mech 1970;37:1031–6.
[13] Reddy JN. A simple higher-order theory for laminated composite plates. J Appl Mech 1984;51(84):745–52.
[14] Shimpi RP, Patel HG. A two variable refined plate theory for orthotropic plate analysis. Int J Solids Struct 2006;43:6783–99.
[15] Ferreira AJM, Fasshauer GE, Batra RC, Rodrigues JD. Static deformations and vibration analysis of composite and sandwich plates using a layerwise theory and RBF-PS discretizations with optimal shape parameter. Compos Struct 2008;86:328–43.
[16] Mindlin RD, Schacknow A, Deresiewicz H. Flexural vibrations of rectangular plates. ASME J Appl Mech 1956;23:430–6.
[17] Brunelle EJ. Buckling of transversely isotropic Mindlin plates. AIAA J 1971;9(6):1018–22.
[18] Shahrokh HH, Arsanjani M. Exact characteristic equations for some of classical boundary conditions of vibrating moderately thick rectangular plates. Int J Solids Struct 2005;42:819–53.
[19] Xing YF, Liu B. Characteristic equations and closed-form solutions for free vibrations of rectangular Mindlin plates. Acta Mech Solid Sin 2009;22(2):125–36.
[20] Xing YF, Liu B. Closed form solutions for free vibrations of rectangular Mindlin plates. Acta Mech Sin 2009;25:689–98.
[21] Srinivas S, Rao AK. Bending, vibration and buckling of simply supported thick orthotropic rectangular plates and laminates. Int J Solids Struct 1970;6:1463–81.
[22] Chung JH, Chung TY, Kim KC. Vibration analysis of orthotropic Mindlin plates with edges elastically restrained against rotation. J Sound Vib 1993;163(1):151–63.
[23] Liew KM, Xiang Y, Kitipornchai S. Research on thick plate vibration: a literature survey. J Sound Vib 1995;180:163–76.
[24] Reddy JN. A review of the literature on finite element modeling of laminated composite plates and shells. Shock Vib Dig 1985;17(4):3–8.
[25] Magrab EB. Natural frequencies of elastically supported orthotropic rectangular plates. J Acoust Soc Am 1977;61:79–83.
[26] Gorman DJ, Ding W. Accurate free vibration analysis of completely free symmetric cross-ply rectangular laminated plates. Compos Struct 2003;60:359–65.
[27] Khdeir AA. Free vibration of antisymmetric angle-ply laminated plates including various boundary conditions. J Sound Vib 1988;122:377–88.
[28] Khdeir AA. Free vibration and buckling of unsymmetric cross-ply laminated plates using a refined theory. J Sound Vib 1989;128:377–95.
[29] Xiang Y, Liew KM, Kitipornchai S. Exact buckling solutions for composite laminates: proper free edge conditions under in-plane loadings. Acta Mech 1996;117:115–28.
[30] Xing YF, Liu B. New exact solutions for free vibrations of rectangular thin plates by symplectic dual method. Acta Mech Sin 2009;25:265–70.
[31] Xing YF, Liu B. A differential quadrature finite element method. Int J Appl Mech 2010;2(1):207–27.
[32] Bhaskar K, Sivaram A. Untruncated infinite series superposition method for accurate flexural analysis of isotropic/orthotropic rectangular plates with arbitrary edge conditions. Compos Struct 2008;83:83–92.
[33] Lam KY, Wang CM, He XQ. Canonical exact solutions for Levy-plates on two-parameter foundation using Green's functions. Eng Struct 2000;22:364–78.