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Exact characteristic equations for free vibrations of thin orthotropic circular cylindrical shells

B. Liu [a,*], Y.F. Xing [a], M.S. Qatu [b], A.J.M. Ferreira [c]

[a] Institute of Solid Mechanics, Beihang University (BUAA), Beijing 100191, China
[b] School of Engineering and Technology, Central Michigan University, Mount Pleasant, MI 48859, United States
[c] Departamento de Engenharia Mecânica e Gestão Industrial, Faculdade de Engenharia da Universidade do Porto, Rua Dr. Roberto Frias, 4200-465 Porto, Portugal

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ABSTRACT

This paper presents an analytical procedure and closed-form vibration solutions with analytically determined coefficients for orthotropic circular cylindrical shells having classical boundary conditions. This analysis is based upon the Donnell–Mushtari shell theory. This is the simplest thin shell theory and its results for the lowest frequencies of a closed cylinder may not be as accurate. It is known that the exact procedure is complicated for orthotropic shells and this complexity has apparently prevented most researchers from getting results. Using the separation of variables method, the closed-form natural frequencies are successfully obtained in this work. They are found in a compact form. Moreover, the characteristics of the eigenvalues are examined. The exact solutions are validated through numerical comparisons with available solutions in literatures and the semi-analytical differential quadrature finite element method (S-DQFEM) solutions calculated by the authors.

1. Introduction

Laminated composite shells are increasingly being used in various engineering applications including aerospace, mechanical, marine, and automotive engineering [1]. The thin-walled circular cylindrical laminated shell has long been an important structural component due to its high stiffness-to-weight and strength-to-weight ratios, and flexibility in design [2,3]. It is, therefore, not surprising that a sizeable number of investigations has been concerned with the vibration behavior of anisotropic or, as a special but important case, orthotropic shells [4]. Of all existing shell models, the closed circular cylindrical shell is perhaps the most widely studied [5–7]. A monograph on vibration of shells has been done by Leissa [6] which contained approximately 1000 references. Of these, more than half deal with circular cylindrical shells. It is very important for engineers to understand the vibration behavior of such shell structures for more reliable and cost effective designs [8].

The problem of calculating the free vibration characteristics of circular cylindrical shells of finite length has been of interest to engineers and scientists for about one and half a century [1,9–11]. The research on shells and their dynamic behavior has been extensively carried out during 1960s and 1970s [6], and has been expanded further in the past four decades [1,12]. Because the equations of motion of cylindrical shells together with relevant boundary conditions are more complex than those of beams and plates, it is very difficult to obtain an analytical solution [13]. The main difficulty lies not in the formulation of a set of equations describing the vibrations of the shell, but rather in the simplification and solution of these equations. In most papers using the exact solutions, only shells with all four boundaries having the shear diaphragm (SD) condition are used [6,11,12,14]. The complexity of the solutions resulting for two opposite edges having SD boundaries, with the others arbitrary, prevented most researchers from getting results [1,6].

If possible and practical, the equations describing the free vibration of a cylindrical shell should be solved exactly because it is difficult to assess the degree with which an approximate solution approximates the exact solution [15]. With the exception of Forsberg [16], exact solutions of these equations have been obtained only after various simplifications have been imposed [17]. Such simplifications limit the range of dimensions or modes for which the solutions are effective [9,18,19]. Forsberg [16] has presented results obtained from an exact solution of the basic differential equations of motion for isotropic circular cylindrical shells. However, the method leads to an eighth-order algebraic equation and an eighth-order frequency determinant which are coupled together. The simultaneous solution of these two systems of equations involves extremely laborious computation as pointed out by Leissa [6] and Li [14]. Although the method requires numerical computation, the results are exact in the same sense that the numerical solution to the transcendental frequency equation for a beam yields an exact solution.

The method of Forsberg [16], and Smith and Haft [17] have an inherent disadvantage in their solution method which centers on the treatment of the arbitrary constants in the expressions for

* Corresponding author.
E-mail address: liubo68@buaa.edu.cn (B. Liu).

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the assumed solution, and the roots of the auxiliary equation. Smith and Haft's and Forsberg's treatment requires that these constants, which are in general complex quantities, be replaced by real ones formulated in a specific manner, which imposes the restriction that the form of the roots of the auxiliary equation must be known before the real constants can be defined. Should the form of these roots change during the solution process, the newly defined constants would no longer be valid. Therefore, it is necessary to continuously monitor the form of these roots throughout the solution process. These disadvantages are overcome by Vronay and Smith [10] by carrying all complex quantities through the solution process as complex. The eight roots of the auxiliary equation, the arbitrary constants of the assumed solution, and the elements of the determinant resulting from the application of boundary conditions are all found to be complex and are treated as such, which leads to a more truly "general" solution.

Due to the effort required to solve for the coefficients of the general solution for the modal displacement, most researchers have resorted to finite element solutions instead of exact solutions. Callahan and Baruh [5] presented a systematic procedure for obtaining the closed-form eigensolution for thin circular cylindrical shell vibrations, which used the software packages MAPLE® and MATLAB® to generate the required expressions and iterate through a natural frequency band to find all associated zeros of the frequency equation. The advantage of explicit displacements over finite element solutions is the absence of numerical error often introduced when derivatives are computed from finite difference approximations.

The methods used for studying the free vibrations of orthotropic circular cylindrical shells include analytical approaches, as well as numerical procedures. Exact solutions for the problem are limited to simply supported shells [4,20–24]. Approximate analytical methods have been presented by Dong [21] and Gulgazaryan et al. [25] for free vibrations of orthotropic circular cylindrical shells with combinations of simply-supported, clamped, and free boundary conditions. The most widely used numerical procedure in free vibration analyses of orthotropic circular cylindrical shells is the Rayleigh–Ritz method [2,26–29]. A semi-analytical finite element method have been used by Sivadas and Ganesan [30] and Ganesan and Sivadas [31] to determine the free vibration characteristics of thin orthotropic circular cylindrical shells with combinations of clamped, free, and simply supported boundary conditions.

In this work, exact characteristic equations and functions are derived for free vibrations of thin orthotropic circular cylindrical shells with several combinations of classical boundary conditions, instead of calculating the coefficients of the general solution through numerical method as Callahan and Baruh [5]. They are found to be in compact and neat forms. The analyses are based on the Donnell–Mushtari shell theory, which is the simplest and most widely used shell theory in thin circular cylindrical shell vibrations [32]. Numerical comparison is carried out with solutions available in literature and obtained by the authors through a highly accurate semi-analytical differential quadrature finite element method (S-DQFEM) to verify the exact solutions. The characteristics of the eigenvalues are studied and it is found that two of the four eigenvalues are very large compared with the other two and can be expressed by the shell thickness and stiffness ratio.

Beside the advantage of explicit displacements over finite element solutions [5], analytic solutions of vibration problems are useful because the results are immediate, suitable for parametric studies, and can serve as benchmarks for numerical methods [33]. There is renewed interest in such classical solutions because the solution methodologies are often applicable with minor changes to modern structures such as the buckling and vibration of nanotubes [34].

The organization of this paper is as follows. In Section 2, the governing differential equations and boundary conditions for free vibrations of thin orthotropic circular cylindrical shells are presented. In Section 3, the characteristic equations and eigenfunctions for the shells with several combinations of simply supported, clamped and free boundary conditions are derived. In Section 4, numerical comparison studies are carried out and the characteristics of the eigenvalues of the shell are analyzed. Finally, conclusions are outlined in Section 5.

2. Governing differential equations and boundary conditions

Consider a circular cylindrical shell element having a constant thickness h, an axial length a, a circumferential length b and a middle surface radius R (Fig. 1). The axial, circumferential and normal to the middle surface coordinate length parameters are denoted with α, β, and ζ, respectively, whereas $u(\alpha,\beta,t)$, $v(\alpha,\beta,t)$ and $w(\alpha,\beta,t)$ represent the corresponding displacement components, where $\beta = R\phi$ and ϕ is the circumferential coordinate. The strain–displacement relations for thin circular cylindrical shells are

$$\left.\begin{array}{lll}\varepsilon_1 = \frac{\partial u}{\partial \alpha}, & \varepsilon_2 = \frac{\partial v}{\partial \beta} + \frac{w}{R}, & \varepsilon_{12} = \frac{\partial u}{\partial \beta} + \frac{\partial v}{\partial \alpha} \\ \chi_1 = -\frac{\partial^2 w}{\partial \alpha^2}, & \chi_2 = -\frac{\partial^2 w}{\partial \beta^2}, & \chi_{12} = -2\frac{\partial^2 w}{\partial \alpha \partial \beta}\end{array}\right\} \quad (1)$$

The definitions for the stress resultants and moments for thin circular cylindrical shells are given by

$$\left.\begin{array}{lll}N_1 = A_{11}\varepsilon_1 + A_{12}\varepsilon_2, & N_2 = A_{22}\varepsilon_2 + A_{21}\varepsilon_1, & S = A_{66}\varepsilon_{12} \\ M_1 = D_{11}\chi_1 + D_{12}\chi_2, & M_2 = D_{22}\chi_2 + D_{21}\chi_1, & M_{12} = D_{66}\chi_{12}\end{array}\right\} \quad (2)$$

where

$$A_{11} = \frac{E_x h}{1-\upsilon_x\upsilon_y},\quad A_{12} = \frac{\upsilon_x E_x h}{1-\upsilon_x\upsilon_y},\quad A_{22} = \frac{E_y h}{1-\upsilon_x\upsilon_y},\quad A_{21} = \frac{\upsilon_y E_y h}{1-\upsilon_x\upsilon_y} \quad (3)$$

are the stretch stiffness and $A_{66} = G_{xy}h$ is the shear stiffness,

$$D_{ij} = kA_{ij}, \qquad i,j = 1,2,6 \quad (4)$$

are the bending and shear rigidities and $k = h^2/12$. In view of the Betti Principle, the product $\upsilon_x E_x = \upsilon_y E_y$. Therefore $A_{12} = A_{21}$ and $D_{12} = D_{21}$. The equations of motion are

$$\left.\begin{array}{l}\frac{\partial N_1}{\partial \alpha} + \frac{\partial S}{\partial \beta} = \rho h \frac{\partial^2 u}{\partial t^2} \\ \frac{\partial N_2}{\partial \beta} + \frac{\partial S}{\partial \alpha} = \rho h \frac{\partial^2 v}{\partial t^2} \\ -\frac{N_2}{R} + \frac{\partial Q_1}{\partial \alpha} + \frac{\partial Q_2}{\partial \beta} = \rho h \frac{\partial^2 w}{\partial t^2}\end{array}\right\} \quad (5)$$

where Q_1 and Q_2 are defined by

$$Q_1 = \frac{\partial M_1}{\partial \alpha} + \frac{\partial M_{12}}{\partial \beta}, \quad Q_2 = \frac{\partial M_2}{\partial \beta} + \frac{\partial M_{12}}{\partial \alpha} \quad (6)$$

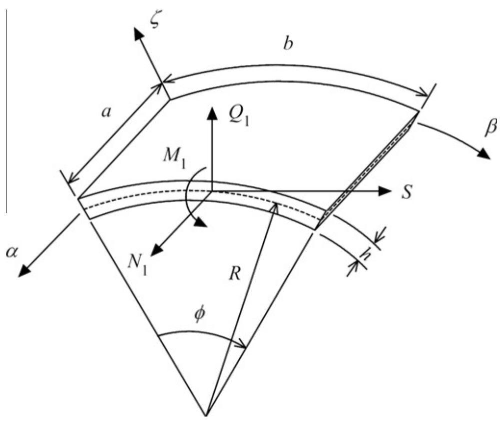

Fig. 1. A circular cylindrical shell element and coordinates.

For free harmonic vibrations, it is assumed that

$$u(\alpha,\beta,t)=u(\alpha,\beta)e^{i\omega t},\quad v(\alpha,\beta,t)=v(\alpha,\beta)e^{i\omega t},\quad w(\alpha,\beta,t)=w(\alpha,\beta)e^{i\omega t} \tag{7}$$

From the above equations, one can easily obtain the Donnell's characteristic equations of free vibration that are used most widely in thin circular cylindrical shell vibrations [32]:

$$\begin{bmatrix} A_{11}\frac{\partial^2}{\partial\alpha^2}+A_{66}\frac{\partial^2}{\partial\beta^2}+\rho h\omega^2 & (A_{12}+A_{66})\frac{\partial^2}{\partial\alpha\partial\beta} & A_{12}\frac{1}{R}\frac{\partial}{\partial\alpha} \\ (A_{12}+A_{66})\frac{\partial^2}{\partial\alpha\partial\beta} & A_{22}\frac{\partial^2}{\partial\beta^2}+A_{66}\frac{\partial^2}{\partial\alpha^2}+\rho h\omega^2 & A_{22}\frac{1}{R}\frac{\partial}{\partial\beta} \\ A_{12}\frac{1}{R}\frac{\partial}{\partial\alpha} & A_{22}\frac{1}{R}\frac{\partial}{\partial\beta} & \bar{L}_{33} \end{bmatrix}\begin{bmatrix} u \\ v \\ w \end{bmatrix}=\mathbf{0} \tag{8}$$

where

$$\bar{L}_{33}=A_{22}\frac{1}{R^2}+k\left[A_{11}\frac{\partial^4}{\partial\alpha^4}+2(A_{12}+2A_{66})\frac{\partial^2}{\partial\alpha^2\partial\beta^2}+A_{22}\frac{\partial^4}{\partial\beta^4}\right]-\rho h\omega^2 \tag{9}$$

Eq. (8) can also be written as

$$\begin{bmatrix} L_{11} & L_{12} & L_{13} \\ & L_{22} & L_{23} \\ \text{symmetric} & & L_{33} \end{bmatrix}\begin{bmatrix} u \\ v \\ w \end{bmatrix}=\mathbf{0} \tag{10a}$$

or

$$\begin{bmatrix} a_1\frac{\partial^2}{\partial\alpha^2}+\frac{\partial^2}{\partial\beta^2}+\gamma^2 & a_3\frac{\partial^2}{\partial\alpha\partial\beta} & a_{12}\frac{1}{R}\frac{\partial}{\partial\alpha} \\ a_3\frac{\partial^2}{\partial\alpha\partial\beta} & a_2\frac{\partial^2}{\partial\beta^2}+\frac{\partial^2}{\partial\alpha^2}+\gamma^2 & a_2\frac{1}{R}\frac{\partial}{\partial\beta} \\ a_{12}\frac{1}{R}\frac{\partial}{\partial\alpha} & a_2\frac{1}{R}\frac{\partial}{\partial\beta} & L_{33} \end{bmatrix}\begin{bmatrix} u \\ v \\ w \end{bmatrix}=\mathbf{0} \tag{10b}$$

where

$$L_{33}=a_2\frac{1}{R^2}+k\left[a_1\frac{\partial^4}{\partial\alpha^4}+2(a_{12}+2)\frac{\partial^2}{\partial\alpha^2\partial\beta^2}+a_2\frac{\partial^4}{\partial\beta^4}\right]-\gamma^2 \tag{11}$$

$$a_1=\frac{A_{11}}{A_{66}},\quad a_2=\frac{A_{22}}{A_{66}},\quad a_{12}=\frac{A_{12}}{A_{66}},\quad a_3=a_{12}+1,\quad \gamma=\omega\sqrt{\frac{\rho h}{A_{66}}} \tag{12}$$

It is well known that there are a number of different boundary conditions associated with a circular cylindrical shell. In this paper, three typical boundary conditions, namely, simple support or shear diaphragm edge, clamped edge and free edge are considered. They are defined as follows:

(1) Simple support or shear diaphragm edge (S), where the flexural deflections, normal moment of flexion, normal stress and tangential displacement are assigned zero values. For $\alpha = a$, they can be expressed in terms of displacements as

$$(v)_{\alpha=a}=0,\quad \left(\frac{\partial u}{\partial\alpha}\right)_{\alpha=a}=0,\quad (w)_{\alpha=a}=0,\quad \left(\frac{\partial^2 w}{\partial\alpha^2}\right)_{\alpha=a}=0 \tag{13}$$

(2) Clamped edge (C), where displacements and rotation on the boundary are assigned zero values. For $\alpha = a$, we have

$$(u)_{\alpha=a}=0,\quad (v)_{\alpha=a}=0,\quad (w)_{\alpha=a}=0,\quad \left(\frac{\partial w}{\partial\alpha}\right)_{\alpha=a}=0 \tag{14}$$

(3) Free edge (F), where stress resultants are assigned zeros values, that is,

$$N_n=0,\quad S+\frac{M_{ns}}{R}=0,\quad M_n=0,\quad Q_n+\frac{\partial M_{ns}}{\partial s}=0 \tag{15a}$$

where n denotes the normal of the edge and s denotes the tangent of the edge. For $\alpha = a$, they can be expressed by displacements as

$$\left[\frac{\partial u}{\partial\alpha}+\upsilon_x\left(\frac{\partial v}{\partial\beta}+\frac{w}{R}\right)\right]_{\alpha=a}=0,\quad \left[\frac{\partial u}{\partial\beta}+\frac{\partial v}{\partial\alpha}-\frac{h^2}{6R}\frac{\partial^2 w}{\partial\alpha\partial\beta}\right]_{\alpha=a}=0$$
$$\left(\frac{\partial^2 w}{\partial\alpha^2}+\upsilon_x\frac{\partial^2 w}{\partial\beta^2}\right)_{\alpha=a}=0,\quad \left[a_1\frac{\partial^3 w}{\partial\alpha^3}+(a_{12}+4)\frac{\partial^3 w}{\partial\alpha\partial\beta^2}\right]_{\alpha=a}=0 \tag{15b}$$

In the next section, the governing differential equations (10) will be solved exactly for closed circular cylindrical shells with several combinations of the above boundary conditions.

3. Exact solutions

This section aims at solving exact solutions for free vibration of circular cylindrical shells. In order to accomplish this task, we will first express u and v by w. This can be easily derived from Eqs. (10), that is

$$L_1u-K\frac{\partial w}{\partial\alpha}=0,\quad L_2v-K\frac{\partial w}{\partial\beta}=0,\quad \frac{1}{R}\left(a_{12}\frac{\partial u}{\partial\alpha}+a_2\frac{\partial v}{\partial\beta}\right)+L_{33}w=0 \tag{16a,b,c}$$

where

$$L_1=a_2L_{11}-a_{12}a_3\frac{\partial^2}{\partial\alpha^2} \tag{17a}$$

$$L_2=a_{12}L_{22}-a_2a_3\frac{\partial^2}{\partial\beta^2} \tag{17b}$$

$$K=Ra_3L_{33}-a_{12}a_2\frac{1}{R} \tag{17c}$$

The substitution of Eqs. (16a,b) into Eq. (16c) leads to

$$\left[\frac{K}{R}\left(a_{12}L_2\frac{\partial^2}{\partial\alpha^2}+a_2L_1\frac{\partial^2}{\partial\beta^2}\right)+L_1L_2L_{33}\right]X=0 \tag{18}$$

where X can be u, v, or w. One can obtain the eigenvalue algebraic equation from Eq. (18).

3.1. Circular cylindrical shells with two simple support edges

The exact solutions for closed circular cylindrical shells with two simple support edges are straightforward. The eigenfunctions u, v, and w can be assumed as

$$\left.\begin{aligned} u(\alpha,\beta)&=\hat{A}\cos(\hat{\alpha}\alpha)\cos(\hat{\beta}\beta) \\ v(\alpha,\beta)&=\hat{B}\sin(\hat{\alpha}\alpha)\sin(\hat{\beta}\beta) \\ w(\alpha,\beta)&=\hat{C}\sin(\hat{\alpha}\alpha)\cos(\hat{\beta}\beta) \end{aligned}\right\} \tag{19a,b,c}$$

where $\hat{\alpha}=m\pi/a$ and $\hat{\beta}=n/R$. Substitution of expressions (19) into Eq. (18) yields

$$\gamma^6+\hat{a}\gamma^4+\hat{b}\gamma^2+\hat{c}=0 \tag{20}$$

where

$$\hat{a}=\frac{s_1-a_{12}a_3\hat{\alpha}^2}{a_2}+\frac{s_2-a_2a_3\hat{\beta}^2}{a_{12}}-s_4$$
$$\hat{b}=\frac{s_3\left(a_{12}^2\hat{\alpha}^2+a_2^2\hat{\beta}^2\right)}{Ra_{12}a_2}-\frac{a_3s_2\hat{\alpha}^2+s_1s_4}{a_2}-\frac{a_3s_1\hat{\beta}^2+s_2s_4}{a_{12}}+\frac{s_1s_2}{a_{12}a_2}$$
$$\hat{c}=\frac{s_2(a_{12}s_3\hat{\alpha}^2-Rs_1s_4)}{Ra_{12}a_2}+\frac{s_1s_3\hat{\beta}^2}{Ra_{12}} \tag{21}$$

where

$$s_1 = \hat{\alpha}^2(a_{12}a_3 - a_1a_2) - a_2\hat{\beta}^2$$
$$s_2 = -a_{12}\hat{\alpha}^2 + a_2\hat{\beta}^2$$
$$s_3 = a_3Rs_4 - \frac{a_{12}a_2}{R} \tag{22}$$
$$s_4 = \frac{a_2}{R^2} + k\left[a_1\hat{\alpha}^4 + a_2\hat{\beta}^4 + 2\hat{\alpha}^2\hat{\beta}^2(a_{12}+2)\right]$$

Eq. (20) is a third-order polynomial in γ^2. Closed-form roots for third- and fourth-order polynomials are available but the formulations are complicated. Hence people have often resorted to approximate techniques at this point. The zeros of a polynomial are the same as the eigenvalues of its companion matrix. This technique is much simpler than the searching techniques and is adopted in this paper. The companion matrix of a polynomial $c_1x^n + c_2x^{n-1} + \cdots + c_nx + c_{n+1}$ is

$$\mathbf{B} = \begin{bmatrix} b_1 & b_2 & \cdots & b_{n-1} & b_n \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \tag{23}$$

where $b_j = -c_{j+1}/c_1$ $(j = 1, 2, \ldots, n)$.

The substitution of Eqs. (19) into Eqs. (16a,b) yields

$$\widehat{A} = \frac{\hat{\alpha}(s_3 - Ra_3\gamma^2)}{s_1 + a_2\gamma^2}\widehat{C}, \quad \widehat{B} = -\frac{\hat{\beta}(s_3 - Ra_3\gamma^2)}{s_2 + a_{12}\gamma^2}\widehat{C} \tag{24}$$

Note that three different frequency parameters can be solved from Eq. (20) for the same $\hat{\alpha}$ and $\hat{\beta}$, but as can be seen from Eq. (24), they correspond to different mode shapes.

3.2. Circular cylindrical shells with α = 0 and a arbitrary

To substitute the separation of variables solution $w = e^{\mu\alpha}e^{\lambda\beta}$, where $\lambda = i\Lambda$, $i = \sqrt{-1}$, $\Lambda = n/R$, n is the number of half-waves in the β direction, into Eq. (18) yields

$$\bar{a}\mu^8 + \bar{b}\mu^6 + \bar{c}\mu^4 + \bar{d}\mu^2 + \bar{e} = 0 \tag{25}$$

where

$$\bar{a} = ka_1^2$$
$$\bar{b} = a_1\left[t_3 + k(t_1 + a_1t_2 + a_3^2\Lambda^2)\right]$$
$$\bar{c} = \left(a_3^2\Lambda^2 + t_1 + a_1t_2\right)t_3 + a_1t_4 + a_1t_1t_2k - \frac{a_{12}}{R^2}$$
$$\bar{d} = \left[\left(a_3^2\Lambda^2 + t_1 + a_1t_2\right)t_4 + t_1t_2t_3\right] - \frac{(2a_{12}a_3 - a_1)a_3^2\Lambda^2 + t_2a_{12}^2}{R^2}$$
$$\bar{e} = t_1\left[t_2t_4 + \frac{a_3^2\Lambda^2}{R^2}\right] \tag{26}$$

where

$$t_1 = \gamma^2 - \Lambda^2$$
$$t_2 = \gamma^2 - a_2\Lambda^2$$
$$t_3 = -2(a_{12}+2)k\Lambda^2 \tag{27}$$
$$t_4 = \frac{a_2}{R^2} + a_2k\Lambda^4 - \gamma^2$$

Assume the roots of Eq. (25) as

$$\lambda_{1,2} = \pm i\,\bar{\alpha}_1, \quad \lambda_{3,4} = \pm i\,\bar{\alpha}_2, \quad \lambda_{5,6} = \pm i\,\bar{\alpha}_3, \quad \lambda_{7,8} = \pm i\,\bar{\alpha}_4 \tag{28}$$

Here $\bar{\alpha}_j$ $(j = 1, 2, 3, 4)$ are real, imaginary or complex. They are arranged by increasing sequence of the absolute values. Then the eigenfunctions u, v, and w in separation of variables form can be expressed in terms of the eigenvalues by four potentials $\overline{W}_j$ $(j = 1, 2, 3, 4)$ as

$$u = \sum_{j=1}^{4}\bar{p}_j\frac{\partial\overline{W}_j}{\partial\alpha}, \quad v = \sum_{j=1}^{4}\bar{q}_j\frac{\partial\overline{W}_j}{\partial\beta}, \quad w = \sum_{j=1}^{4}\overline{W}_j \tag{29a, b, c}$$

where

$$\overline{W}_j(\alpha,\beta) = (\overline{A}_j\sin\bar{\alpha}_j\alpha + \overline{B}_j\cos\bar{\alpha}_j\alpha)\cos\Lambda\beta \tag{30}$$

The substitution of expressions (29) into Eqs. (16a,b) leads to

$$\bar{p}_j = \frac{\overline{C}_j}{a_{12}a_3\bar{\alpha}_j^2 + a_2(t_1 - a_1\bar{\alpha}_j^2)}, \quad \bar{q}_j = \frac{\overline{C}_j}{a_2a_3\Lambda^2 + a_{12}(t_2 - \bar{\alpha}_j^2)} \tag{31}$$

where

$$\overline{C}_j = Ra_3\left(a_1k\bar{\alpha}_j^4 - t_3\bar{\alpha}_j^2 + t_4\right) - \frac{a_{12}a_2}{R} \tag{32}$$

To substitute expressions (29)–(32) into Eq. (16c) leads to

$$\bar{a}\bar{\alpha}_j^8 - \bar{b}\bar{\alpha}_j^6 + \bar{c}\bar{\alpha}_j^4 - \bar{d}\bar{\alpha}_j^2 + \bar{e} = 0, \quad j = 1,2,3,4 \tag{33}$$

The correctness of Eq. (33) is straightforward since we solved $\bar{\alpha}_j$ $(j = 1, 2, 3, 4)$ from Eq. (25), in other words, the exactness of Eqs. (29) is proved.

For circular cylindrical shells with α = 0, a arbitrary, the exact characteristic equations and eigenfunctions are obtained by substituting expressions (29) into boundary conditions. The characteristic equations for the cases S–C and S–F can be given as follows:

Case S–C:

$$\begin{aligned}&\bar{\alpha}_1\bar{\alpha}_2(\bar{p}_1-\bar{p}_2)(\bar{q}_4-\bar{q}_3)\cos(a\bar{\alpha}_1)\cos(a\bar{\alpha}_2)\sin(a\bar{\alpha}_3)\sin(a\bar{\alpha}_4)\\&\quad+\bar{\alpha}_1\bar{\alpha}_3(\bar{p}_1-\bar{p}_3)(\bar{q}_2-\bar{q}_4)\cos(a\bar{\alpha}_1)\sin(a\bar{\alpha}_2)\cos(a\bar{\alpha}_3)\\&\quad\times\sin(a\bar{\alpha}_4)+\bar{\alpha}_1\bar{\alpha}_4(\bar{p}_1-\bar{p}_4)(\bar{q}_3-\bar{q}_2)\cos(a\bar{\alpha}_1)\\&\quad\times\sin(a\bar{\alpha}_2)\sin(a\bar{\alpha}_3)\cos(a\bar{\alpha}_4)+\bar{\alpha}_2\bar{\alpha}_3(\bar{p}_2-\bar{p}_3)(\bar{q}_4-\bar{q}_1)\\&\quad\times\sin(a\bar{\alpha}_1)\cos(a\bar{\alpha}_2)\cos(a\bar{\alpha}_3)\sin(a\bar{\alpha}_4)+\bar{\alpha}_2\bar{\alpha}_4(\bar{p}_2-\bar{p}_4)\\&\quad\times(\bar{q}_1-\bar{q}_3)\sin(a\bar{\alpha}_1)\cos(a\bar{\alpha}_2)\sin(a\bar{\alpha}_3)\cos(a\bar{\alpha}_4)\\&\quad+\bar{\alpha}_3\bar{\alpha}_4(\bar{p}_3-\bar{p}_4)(\bar{q}_2-\bar{q}_1)\sin(a\bar{\alpha}_1)\sin(a\bar{\alpha}_2)\\&\quad\times\cos(a\bar{\alpha}_3)\cos(a\bar{\alpha}_4)=0\end{aligned} \tag{34}$$

Case S–F:

$$\begin{aligned}&(d_1e_2-d_2e_1)(f_3k_4-f_4k_3)\sin(a\bar{\alpha}_1)\sin(a\bar{\alpha}_2)\cos(a\bar{\alpha}_3)\cos(a\bar{\alpha}_4)\\&\quad+(d_1e_3-d_3e_1)(f_4k_2-f_2k_4)\sin(a\bar{\alpha}_1)\cos(a\bar{\alpha}_2)\\&\quad\times\sin(a\bar{\alpha}_3)\cos(a\bar{\alpha}_4)+(d_1e_4-d_4e_1)(f_2k_3-f_3k_2)\\&\quad\times\sin(a\bar{\alpha}_1)\cos(a\bar{\alpha}_2)\cos(a\bar{\alpha}_3)\sin(a\bar{\alpha}_4)+(d_2e_3-d_3e_2)\\&\quad\times(f_1k_4-f_4k_1)\cos(a\bar{\alpha}_1)\sin(a\bar{\alpha}_2)\sin(a\bar{\alpha}_3)\cos(a\bar{\alpha}_4)\\&\quad+(d_2e_4-d_4e_2)(f_3k_1-f_1k_3)\cos(a\bar{\alpha}_1)\sin(a\bar{\alpha}_2)\\&\quad\times\cos(a\bar{\alpha}_3)\sin(a\bar{\alpha}_4)+(d_3e_4-d_4e_3)(f_1k_2-f_2k_1)\\&\quad\times\cos(a\bar{\alpha}_1)\cos(a\bar{\alpha}_2)\sin(a\bar{\alpha}_3)\sin(a\bar{\alpha}_4)=0\end{aligned} \tag{35}$$

where

$$d_j = \upsilon_x\left(\frac{1}{R} - \bar{q}_j\Lambda^2\right) - \bar{p}_j\bar{\alpha}_j^2$$
$$e_j = -\upsilon_x\Lambda^2 - \bar{\alpha}_j^2$$
$$f_j = \left(\frac{h^2}{6R} - \bar{q}_j - \bar{p}_j\right)\Lambda\bar{\alpha}_j \tag{36a, b, c, d}$$
$$k_j = -\left[(a_{12}+4)\Lambda^2 + a_1\bar{\alpha}_j^2\right]\bar{\alpha}_j$$

where $j = 1, 2, 3, 4$.

For the cases C–C and F–F, the mode shapes are symmetric or anti-symmetric. Thus the characteristic equations can be simplified. They are

Case C–C (symmetric):

$$\begin{aligned}&\bar{\alpha}_1\bar{\alpha}_2(\bar{p}_1-\bar{p}_2)(\bar{q}_4-\bar{q}_3)\sin(\tilde{a}\bar{\alpha}_1)\sin(\tilde{a}\bar{\alpha}_2)\cos(\tilde{a}\bar{\alpha}_3)\\&\quad\times\cos(\tilde{a}\bar{\alpha}_4)+\bar{\alpha}_1\bar{\alpha}_3(\bar{p}_1-\bar{p}_3)(\bar{q}_2-\bar{q}_4)\sin(\tilde{a}\bar{\alpha}_1)\\&\quad\times\cos(\tilde{a}\bar{\alpha}_2)\sin(\tilde{a}\bar{\alpha}_3)\cos(\tilde{a}\bar{\alpha}_4)+\bar{\alpha}_1\bar{\alpha}_4(\bar{p}_1-\bar{p}_4)(\bar{q}_3-\bar{q}_2)\\&\quad\times\sin(\tilde{a}\bar{\alpha}_1)\cos(\tilde{a}\bar{\alpha}_2)\cos(\tilde{a}\bar{\alpha}_3)\sin(\tilde{a}\bar{\alpha}_4)\\&\quad+\bar{\alpha}_2\bar{\alpha}_3(\bar{p}_2-\bar{p}_3)(\bar{q}_4-\bar{q}_1)\cos(\tilde{a}\bar{\alpha}_1)\sin(\tilde{a}\bar{\alpha}_2)\\&\quad\times\sin(\tilde{a}\bar{\alpha}_3)\cos(\tilde{a}\bar{\alpha}_4)+\bar{\alpha}_2\bar{\alpha}_4(\bar{p}_2-\bar{p}_4)(\bar{q}_1-\bar{q}_3)\\&\quad\times\cos(\tilde{a}\bar{\alpha}_1)\sin(\tilde{a}\bar{\alpha}_2)\cos(\tilde{a}\bar{\alpha}_3)\sin(\tilde{a}\bar{\alpha}_4)\\&\quad+\bar{\alpha}_3\bar{\alpha}_4(\bar{p}_3-\bar{p}_4)(\bar{q}_2-\bar{q}_1)\cos(\tilde{a}\bar{\alpha}_1)\cos(\tilde{a}\bar{\alpha}_2)\\&\quad\times\sin(\tilde{a}\bar{\alpha}_3)\sin(\tilde{a}\bar{\alpha}_4)=0\end{aligned}\tag{37}$$

where

$$\tilde{a}=a/2$$

Case F–F (symmetric):

$$\begin{aligned}&(d_1e_2-d_2e_1)(f_3k_4-f_4k_3)\cos(\tilde{a}\bar{\alpha}_1)\cos(\tilde{a}\bar{\alpha}_2)\sin(\tilde{a}\bar{\alpha}_3)\\&\quad\times\sin(\tilde{a}\bar{\alpha}_4)+(d_1e_3-d_3e_1)(f_4k_2-f_2k_4)\cos(\tilde{a}\bar{\alpha}_1)\\&\quad\times\sin(\tilde{a}\bar{\alpha}_2)\cos(\tilde{a}\bar{\alpha}_3)\sin(\tilde{a}\bar{\alpha}_4)+(d_1e_4-d_4e_1)(f_2k_3-f_3k_2)\\&\quad\times\cos(\tilde{a}\bar{\alpha}_1)\sin(\tilde{a}\bar{\alpha}_2)\sin(\tilde{a}\bar{\alpha}_3)\cos(\tilde{a}\bar{\alpha}_4)\\&\quad+(d_2e_3-d_3e_2)(f_1k_4-f_4k_1)\sin(\tilde{a}\bar{\alpha}_1)\cos(\tilde{a}\bar{\alpha}_2)\cos(\tilde{a}\bar{\alpha}_3)\\&\quad\times\sin(\tilde{a}\bar{\alpha}_4)+(d_2e_4-d_4e_2)(f_3k_1-f_1k_3)\sin(\tilde{a}\bar{\alpha}_1)\cos(\tilde{a}\bar{\alpha}_2)\\&\quad\times\sin(\tilde{a}\bar{\alpha}_3)\cos(\tilde{a}\bar{\alpha}_4)+(d_3e_4-d_4e_3)(f_1k_2-f_2k_1)\sin(\tilde{a}\bar{\alpha}_1)\\&\quad\times\sin(\tilde{a}\bar{\alpha}_2)\cos(\tilde{a}\bar{\alpha}_3)\cos(\tilde{a}\bar{\alpha}_4)=0\end{aligned}\tag{38}$$

Note that the origin of coordinates of Eqs. (37) and (38) is the middle point of the axis of the circular cylindrical shells instead of an end of the axis. The characteristic equations for anti-symmetric modes of cases C–C and F–F can be obtained through substitution of $\sin(a\bar{\alpha}_j)$ and $\cos(a\bar{\alpha}_j)$ by $\cos(a\bar{\alpha}_j)$ and $\sin(a\bar{\alpha}_j)$, respectively.

The coefficients of the eigenfunctions for the cases S–C and S–F can be given as follows:

Case S–C:

$$\begin{aligned}&A_1=1,\quad A_2=-\frac{P_1}{P_2},\quad A_3=\frac{P_4}{P_2},\quad A_4=-\frac{P_3}{P_2},\\&B_1=B_2=B_3=B_4=0\end{aligned}\tag{39}$$

where

$$\begin{aligned}P_i=\;&\bar{\alpha}_i\cos(a\bar{\alpha}_i)\sin(a\bar{\alpha}_3)\sin(a\bar{\alpha}_4)(\bar{q}_3-\bar{q}_4)-\bar{\alpha}_3\sin(a\bar{\alpha}_i)\\&\times\cos(a\bar{\alpha}_3)\sin(a\bar{\alpha}_4)(\bar{q}_i-\bar{q}_4)+\bar{\alpha}_4\sin(a\bar{\alpha}_i)\sin(a\bar{\alpha}_3)\\&\times\cos(a\bar{\alpha}_4)(\bar{q}_i-\bar{q}_3)\end{aligned}\tag{40a}$$

$$\begin{aligned}P_j=\;&\bar{\alpha}_1\cos(a\bar{\alpha}_1)\sin(a\bar{\alpha}_2)\sin(a\bar{\alpha}_j)(\bar{q}_2-\bar{q}_j)-\bar{\alpha}_2\sin(a\bar{\alpha}_1)\\&\times\cos(a\bar{\alpha}_2)\sin(a\bar{\alpha}_j)(\bar{q}_1-\bar{q}_j)+\bar{\alpha}_j\sin(a\bar{\alpha}_1)\sin(a\bar{\alpha}_2)\\&\times\cos(a\bar{\alpha}_j)(\bar{q}_1-\bar{q}_2)\end{aligned}\tag{40b}$$

where $i=1,2$; $j=3,4$.

Case S–F: Eq. (39) can be applied to this case directly, but the expressions for P_k, $k=1,2,3,4$ are

$$\begin{aligned}P_i=\;&d_i\sin(a\bar{\alpha}_i)\cos(a\bar{\alpha}_3)\cos(a\bar{\alpha}_4)(e_3f_4-e_4f_3)-d_3\cos(a\bar{\alpha}_i)\\&\times\sin(a\bar{\alpha}_3)\cos(a\bar{\alpha}_4)(e_if_4-e_4f_i)+d_4\cos(a\bar{\alpha}_i)\cos(a\bar{\alpha}_3)\\&\times\sin(a\bar{\alpha}_4)(e_if_3-e_3f_i)\end{aligned}\tag{41a}$$

$$\begin{aligned}P_j=\;&d_1\sin(a\bar{\alpha}_1)\cos(a\bar{\alpha}_2)\cos(a\bar{\alpha}_j)(e_2f_j-e_jf_2)-d_2\cos(a\bar{\alpha}_1)\\&\times\sin(a\bar{\alpha}_2)\cos(a\bar{\alpha}_j)(e_1f_j-e_jf_1)+d_3\cos(a\bar{\alpha}_1)\cos(a\bar{\alpha}_2)\\&\times\sin(a\bar{\alpha}_j)(e_1f_2-e_2f_1)\end{aligned}\tag{41b}$$

where $i=1,2$; $j=3,4$.

The coefficients of the eigenefunctions for the cases C–C and F–F are

Case C–C (symmetric):

$$\begin{aligned}&B_1=1,\quad B_2=-\frac{P_1}{P_2},\quad B_3=\frac{P_4}{P_2},\quad B_4=-\frac{P_3}{P_2},\\&A_1=A_2=A_3=A_4=0\end{aligned}\tag{42}$$

$$\begin{aligned}P_i=\;&\bar{\alpha}_i\cos(\tilde{a}\bar{\alpha}_3)\cos(\tilde{a}\bar{\alpha}_4)\sin(\tilde{a}\bar{\alpha}_i)(\bar{q}_3-\bar{q}_4)+\bar{\alpha}_3\cos(\tilde{a}\bar{\alpha}_i)\\&\times\cos(\tilde{a}\bar{\alpha}_4)\sin(\tilde{a}\bar{\alpha}_3)(\bar{q}_4-\bar{q}_i)+\bar{\alpha}_4\cos(\tilde{a}\bar{\alpha}_i)\cos(\tilde{a}\bar{\alpha}_3)\\&\times\sin(\tilde{a}\bar{\alpha}_4)(\bar{q}_i-\bar{q}_3)\end{aligned}\tag{43a}$$

$$\begin{aligned}P_j=\;&\bar{\alpha}_1\cos(\tilde{a}\bar{\alpha}_2)\cos(\tilde{a}\bar{\alpha}_j)\sin(\tilde{a}\bar{\alpha}_1)(\bar{q}_2-\bar{q}_j)+\bar{\alpha}_2\cos(\tilde{a}\bar{\alpha}_1)\\&\times\cos(\tilde{a}\bar{\alpha}_j)\sin(\tilde{a}\bar{\alpha}_2)(\bar{q}_j-\bar{q}_1)+\bar{\alpha}_j\cos(\tilde{a}\bar{\alpha}_1)\cos(\tilde{a}\bar{\alpha}_2)\\&\times\sin(\tilde{a}\bar{\alpha}_j)(\bar{q}_1-\bar{q}_2)\end{aligned}\tag{43b}$$

where $i=1,2$; $j=3,4$.

Case F–F (symmetric):

To apply Eq. (42) to this case, the expressions for P_k, $k=1,2,3,4$ are

$$\begin{aligned}P_i=\;&e_i\cos(\tilde{a}\bar{\alpha}_i)\sin(\tilde{a}\bar{\alpha}_3)\sin(\tilde{a}\bar{\alpha}_4)(f_3k_4-f_4k_3)+e_3\cos(\tilde{a}\bar{\alpha}_3)\\&\times\sin(\tilde{a}\bar{\alpha}_i)\sin(\tilde{a}\bar{\alpha}_4)(f_4k_i-f_ik_4)+e_4\cos(\tilde{a}\bar{\alpha}_4)\sin(\tilde{a}\bar{\alpha}_i)\\&\times\sin(\tilde{a}\bar{\alpha}_3)(f_ik_3-f_3k_i)\end{aligned}\tag{44a}$$

$$\begin{aligned}P_j=\;&e_1\cos(\tilde{a}\bar{\alpha}_1)\sin(\tilde{a}\bar{\alpha}_2)\sin(\tilde{a}\bar{\alpha}_j)(f_2k_j-f_jk_2)+e_2\cos(\tilde{a}\bar{\alpha}_2)\\&\times\sin(\tilde{a}\bar{\alpha}_1)\sin(\tilde{a}\bar{\alpha}_j)(f_jk_1-f_1k_j)+e_j\cos(\tilde{a}\bar{\alpha}_j)\sin(\tilde{a}\bar{\alpha}_1)\\&\times\sin(\tilde{a}\bar{\alpha}_2)(f_1k_2-f_2k_1)\end{aligned}\tag{44b}$$

The coefficients of the eigenefunctions for anti-symmetric modes of C–C and F–F shells can be expressed as

$$\begin{aligned}&A_1=1,\quad A_2=-\frac{P_1}{P_2},\quad A_3=\frac{P_4}{P_2},\quad A_4=-\frac{P_3}{P_2},\\&B_1=B_2=B_3=B_4=0\end{aligned}\tag{45}$$

The expressions in Eq. (45) for the two cases can be obtained from Eqs. (43) and (44) with substitution of $\sin(a\bar{\alpha}_j)$ and $\cos(a\bar{\alpha}_j)$ by $\cos(a\bar{\alpha}_j)$ and $\sin(a\bar{\alpha}_j)$, respectively.

The exact solutions for clamped-free circular cylindrical shells will not be presented here because they have computational problem and special characteristics, and a single paper cannot include so much content.

4. Results and discussion

In order to study the characteristics of the exact solutions and to obtain benchmark solutions, an efficient and accurate semi-analytical differential quadrature finite element method (S-DQFEM) is developed. The formulations of S-DQFEM are presented in Appendix. Both the accuracy of S-DQFEM and the exact solutions is verified through numerical comparison with the exact solutions of Smith and Haft [17] for a clamped isotropic circular cylindrical shell in Table 1, and the exact solutions of Ganesan and Sivadas [31] and Greenberg and Stavsky [24] for a simply supported orthotropic circular cylindrical shell based on Love's theory in Table 2. The material constants of Tables 1 and 2 are included in the captions. To avoid comparisons with round off errors, the natural frequencies are expressed as the same frequency parameters used by the authors. It may be seen from Tables 1 and 2 that the S-DQFEM solutions agree with the exact solutions for all the significant digits used for comparisons, and these results agree with the results of

Table 1
Frequencies (cps) for a clamped circular cylindrical shell, $a = 12$ in., $R = 3$ in., $h = 0.01$ in., $E = 29.6 \times 10^6$ psi, $\upsilon = 0.29$, $\rho = 0.733 \times 10^{-3}$ lb-s^2/in.^{4}.

$m\backslash n$		1	2	3	4	5	6	7	8
1	Smith–Haft [17]	3427	1918	1145	765	580	538	597	721
	Exact [present]	3425	1917	1154	764	580	538	598	723
	S-DQFEM [app.]	3425	1917	1154	764	580	538	598	723
2	Smith–Haft [17]	6423	3905	2538	1753	1287	1022	907	911
	Exact [present]	6412	3903	2537	1752	1287	1022	907	911
	S-DQFEM [app.]	6412	3903	2537	1752	1287	1022	907	911
3	Smith–Haft [17]	–	5844	4054	2921	2192	1720	1431	1287
	Exact [present]	8493	5841	4052	2920	2191	1720	1431	1287
	S-DQFEM [app.]	8493	5841	4052	2920	2191	1720	1431	1287
4	Smith–Haft [17]	–	7303	5447	4104	3168	2516	2076	1797
	Exact [present]	9420	7299	5444	4102	3167	2518	2077	1797
	S-DQFEM [app.]	9420	7299	5444	4102	3167	2518	2077	1797

Table 2
Circular frequencies (Hz) for a simply supported circular cylindrical shell, $a = 5.0$ m, $R = 1.0$ m, $h = 0.01$ m, $E_x = 120$ GPa, $E_y = 10$ GPa, $G_{xy} = 5.5$ GPa, $\upsilon_y = 0.27$, $\rho = 1700$ kg/m^3, $m = 1$.

n	1	2	3	4	5	6	7	8	9
Exact [present]	741	416	258	198	209	266	350	452	570
S-DQFEM [app.]	741	416	258	198	209	266	350	452	570
Ganesan and Sivadas [31]	743	417	257	194	204	260	344	447	567
Greenberg and Stavsky [24]	765	430	266	202	211	270	358	464	587

Table 3
Comparison of the frequency parameters $\Omega = \pi^2\omega\sqrt{\rho/G_{xy}}$ for closed circular cylindrical shells with S–C boundary conditions ($E_x/E_y = 2$, $h/R = 0.01$).

a/R	(m, n)	$\bar{\alpha}_1 a/\pi$	$\bar{\alpha}_2 a/\pi$	$\bar{\alpha}_3 a/\pi$	$\bar{\alpha}_4 a/\pi$	Exact [present]	S-DQFEM [app.]
2	(1, 1)	$-0.1191i$*	1.0013	$5.9518 + 6.0239i$	$5.9518 - 6.0239i$	9.1888	9.1891
	(2, 1)	0.2700	2.0112	$4.3215 + 4.5738i$	$4.3215 - 4.5738i$	12.4667	12.4669
	(1, 2)	$0.5514i$	1.0358	$6.5201 + 6.7206i$	$6.5201 - 6.7206i$	5.8588	5.8590
	(2, 2)	$-0.5712i$	2.0108	$5.5368 + 5.8827i$	$5.5368 - 5.8827i$	10.1288	10.1290
10	(1, 1)	$0.9324i$	1.1584	$34.6161 + 34.8297i$	$34.6161 - 34.8297i$	1.4662	1.4662
	(2, 1)	$1.2082i$	2.0702	$34.1096 + 34.3338i$	$34.1096 - 34.3338i$	3.5414	3.5415
	(1, 2)	$1.1485i$	1.2181	$34.3871 + 35.2349i$	$34.3871 - 35.2349i$	0.6128	0.6128
	(2, 2)	$1.8230i$	2.1644	$34.2671 + 35.1241i$	$34.2671 - 35.1241i$	1.6824	1.6824
20	(1, 1)	$-1.3184i$	1.2195	$69.4154 + 69.8382i$	$69.4154 - 69.8382i$	0.4685	0.4685
	(2, 1)	$-1.7857i$	2.1675	$69.2730 + 69.6995i$	$69.2730 - 69.6995i$	1.3117	1.3117
	(1, 2)	$-1.2299i$	1.2400	$68.8062 + 70.4993i$	$68.8062 - 70.4993i$	0.2171	0.2171
	(2, 2)	$-2.1176i$	2.2231	$68.7843 + 70.4791i$	$68.7843 - 70.4791i$	0.5214	0.5214

* $i = \sqrt{-1}$.

Ref. [17] for at least three significant digits for most frequencies. The present results also agree well with those of Refs. [24,31] as can be seen from Table 2 although different shell theories have been employed.

Then the exact frequency parameters for free vibrations of orthotropic circular cylindrical shells with several combinations of simple support, clamp, and free edge conditions are presented in Tables 3–10 and the characteristics of the eigenvalues are investigated. In the calculations, we take $\upsilon_y = 0.3$, $a_2 = 2$, $a_3 = (A_{66} + A_{21})/A_{66} = 1 + \upsilon_y a_2 = 1.6$, which are reasonable for many materials. Different length–radius ratio a/R or stiffness ratio $E_x/E_y = a_1/a_2 = a_1/2$ are taken into account, and $\upsilon_x = \upsilon_y E_y/E_x$.

In Tables 3–6, exact frequency parameters are compared with the S-DQFEM solutions for the cases with S–C, S–F, C–C, and F–F boundary conditions, $E_x/E_y = 2$, $h/R = 0.01$, and $a/R = 2, 10, 20$. The non-dimensional eigenvalues $a\bar{\alpha}_j/\pi$ $(j = 1, 2, 3, 4)$ are included in Tables 3 and 4. In Tables 5 and 6, non-dimensional eigenvalues $\tilde{a}\bar{\alpha}_j/\pi$ $(j = 1, 2, 3, 4)$ are included. It can be seen that the exact solutions agree with the S-DQFEM solutions for at least four significant digits for most frequency parameters. One should noted that the magnitudes of $\bar{\alpha}_j$ $(j = 3, 4)$ in Tables 3–6 are much larger than those of $\bar{\alpha}_j$ $(j = 1, 2)$ and seem to be proportional to the length–radius ratio a/R. The eigenvalues $\tilde{a}\bar{\alpha}_j/\pi$ in Tables 5 and 6 are half of the eigenvalues $a\bar{\alpha}_j/\pi$ in Tables 3 and 4 because we used only half $(\tilde{a} = a/2)$ of the circular cylindrical shells for calculations according to the symmetric and anti-symmetric characteristic of the modes of the shells with C–C and F–F boundary conditions.

In order to investigate the relationship between the non-dimensional eigenvalues and the thickness–radius ratios, exact frequency parameters are computed for the orthotropic circular cylindrical shells with C–C and F–F boundary conditions, $E_x/E_y = 2$, $a/R = 10$, $h/R = 0.05, 0.005$ and 0.001. One can see from Tables 7 and 8 that the two larger eigenvalues increase dramatically with the decreasing of the thickness–radius ratio h/R. We infer that there may be a simple relation to reveal the effects of h/R on the eigenvalues. In order to find the relation, the real part of $\bar{\alpha}_3$ for the fundamental frequencies is calculated using $E_x/E_y = 2$, $a/R = 2, 5, 10, 15, 20$, $h/R = 0.1, 0.01, 0.005, 0.002, 0.001$, and C–C and F–F boundary conditions. The results are graphically presented in Fig. 2. One can see that, except for the cases of $a/R = 2$, all other lines are nearly overlapped. Thus, the following equation is true for

Table 4
Comparison of the frequency parameters $\Omega = \pi^2\omega\sqrt{\rho/G_{xy}}$ for closed circular cylindrical shells with S–F boundary conditions (E_x/E_y = 2, h/R = 0.01).

a/R	(m, n)	$\bar{\alpha}_1 a/\pi$	$\bar{\alpha}_2 a/\pi$	$\bar{\alpha}_3 a/\pi$	$\bar{\alpha}_4 a/\pi$	Exact [present]	S-DQFEM [app.]
2	(2, 1)	0.0820i	1.0582	5.8446 + 5.9218i	5.8446 − 5.9218i	9.5583	9.5588
	(3, 1)	0.2241	1.6107	4.8748 + 5.0314i	4.8748 − 5.0314i	11.7727	11.7727
	(2, 2)	0.5724i	1.3381	6.2380 + 6.4695i	6.2380 − 6.4695i	7.5779	7.5785
	(3, 2)	0.5631i	2.3882	5.1566 + 5.5972i	5.1566 − 5.5972i	10.9957	10.9960
10	(2, 1)	−0.9858i	1.2691	34.5802 + 34.7946i	34.5802 − 34.7946i	1.7031	1.7032
	(3, 1)	−1.2285i	2.2929	33.9120 + 34.1404i	33.9120 − 34.1404i	4.0588	4.0590
	(2, 2)	−1.1818i	1.2584	34.3848 + 35.2508i	34.3848 − 35.2508i	0.6493	0.6495
	(3, 2)	−1.8860i	2.2732	34.2423 + 35.1012i	34.2423 − 35.1012i	1.8261	1.8263
20	(2, 1)	−1.1676i	1.2556	69.4130 + 69.8368i	69.4130 − 69.8368i	0.4947	0.4947
	(3, 1)	1.8387i	2.2665	69.2466 + 69.6737i	69.2466 − 69.6737i	1.4134	1.4135
	(2, 2)	1.2430i	1.2539	68.8060 + 70.4992i	68.8060 − 70.4992i	0.2198	0.2200
	(3, 2)	2.1466i	2.2572	68.7828 + 70.4778i	68.7828 − 70.4778i	0.5355	0.5356

Table 5
Comparison of the frequency parameters $\Omega = \pi^2\omega\sqrt{\rho/G_{xy}}$ for closed circular cylindrical shells with C–C boundary conditions (E_x/E_y = 2, h/R = 0.01).

a/R	(m, n)	$\bar{\alpha}_1 \tilde{a}/\pi$	$\bar{\alpha}_2 \tilde{a}/\pi$	$\bar{\alpha}_3 \tilde{a}/\pi$	$\bar{\alpha}_4 \tilde{a}/\pi$	Exact [present]	S-DQFEM [app.]
2	(1, 1)	−0.0558i	0.5073	2.9634 + 3.0001i	2.9634 − 3.0001i	9.2777	9.2780
	(2, 1)	0.1356	1.0128	2.1518 + 2.2800i	2.1518 − 2.2800i	12.4846	12.4847
	(1, 2)	−0.2787i	0.5459	3.2359 + 3.3385i	3.2359 − 3.3385i	6.2057	6.2059
	(2, 2)	0.2855i	1.0105	2.7631 + 2.9372i	2.7631 − 2.9372i	10.1567	10.1570
10	(1, 1)	0.5011i	0.6530	17.2834 + 17.4627i	17.2834 − 17.4627i	1.7838	1.7839
	(2, 1)	−0.6074i	1.0635	17.0309 + 17.1435i	17.0309 − 17.1435i	3.6740	3.6742
	(1, 2)	−0.6578i	0.7127	17.1867 + 17.6111i	17.1867 − 17.6111i	0.8111	0.8112
	(2, 2)	−0.9533i	1.1549	17.1167 + 17.5465i	17.1167 − 17.5465i	1.8751	1.8752
20	(1, 1)	0.6515i	0.7147	34.6994 + 34.9114i	34.6994 − 34.9114i	0.6284	0.6284
	(2, 1)	−0.9324i	1.1584	34.6161 + 34.8297i	34.6161 − 34.8297i	1.4662	1.4663
	(1, 2)	−0.7274i	0.7403	34.4018 + 35.2485i	34.4018 − 35.2485i	0.2709	0.2709
	(2, 2)	−1.1485i	1.2180	34.3871 + 35.2349i	34.3871 − 35.2349i	0.6128	0.6128

Table 6
Comparison of the frequency parameters $\Omega = \pi^2\omega\sqrt{\rho/G_{xy}}$ for closed circular cylindrical shells with F–F boundary conditions (E_x/E_y = 2, h/R = 0.01).

a/R	(m, n)	$\bar{\alpha}_1 \tilde{a}/\pi$	$\bar{\alpha}_2 \tilde{a}/\pi$	$\bar{\alpha}_3 \tilde{a}/\pi$	$\bar{\alpha}_4 \tilde{a}/\pi$	Exact [present]	S-DQFEM [app.]
2	(1, 1)	0.1037	0.7591	2.5105 + 2.5801i	2.5105 − 2.5801i	11.5353	11.5351
	(2, 1)	0.1320	0.9697	2.2061 + 2.3224i	2.2061 − 2.3224i	12.3720	12.3719
	(1, 2)	0.2873i	0.9055	2.8727 + 3.0252i	2.8727 − 3.0252i	9.5300	9.5298
	(2, 2)	−0.2779	1.3672	2.4142 + 2.6883i	2.4142 − 2.6883i	11.5713	11.5713
10	(1, 1)	−0.5510i	0.7869	17.2229 + 17.3314i	17.2229 − 17.3314i	2.3869	2.3867
	(2, 1)	0.6170i	1.2917	16.8062 + 16.9238i	16.8062 − 16.9238i	4.7225	4.7224
	(1, 2)	0.6985i	0.7658	17.1820 + 17.6068i	17.1820 − 17.6068i	0.9220	0.9222
	(2, 2)	−1.0181i	1.2775	17.0828 + 17.5152i	17.0828 − 17.5152i	2.2105	2.2105
20	(1, 1)	0.6871i	0.7625	34.6944 + 34.9066i	34.6944 − 34.9066i	0.7072	0.7072
	(2, 1)	0.9858i	1.2691	34.5802 + 34.7946i	34.5802 − 34.7946i	1.7031	1.7030
	(1, 2)	−0.7432i	0.7574	34.4015 + 35.2483i	34.4015 − 35.2483i	0.2797	0.2800
	(2, 2)	−1.1818	1.2584	34.3848 + 35.2328i	34.3848 − 35.2328i	0.6493	0.6494

Table 7
Comparison of the frequency parameters $\Omega = \pi^2\omega\sqrt{\rho/G_{xy}}$ for closed circular cylindrical shells with C–C boundary conditions (E_x/E_y = 2, a/R = 10).

h/R	(m, n)	$\bar{\alpha}_1 \tilde{a}/\pi$	$\bar{\alpha}_2 \tilde{a}/\pi$	$\bar{\alpha}_3 \tilde{a}/\pi$	$\bar{\alpha}_4 \tilde{a}/\pi$	Exact [present]	S-DQFEM [app.]
0.05	(1, 1)	−0.5018i	0.6540	7.6347 + 7.8747i	7.6347 − 7.8747i	1.7958	1.7959
	(2, 1)	−0.6077i	1.0654	7.5168 + 7.7685i	7.5168 − 7.7685i	3.6894	3.6896
	(1, 2)	0.6779i	0.7071	7.3295 + 8.2743i	7.3295 − 8.2743i	1.0948	1.0949
	(2, 2)	0.9680i	1.1561	7.2930 + 8.2500i	7.2930 − 8.2500i	2.0507	2.0508
0.005	(1, 1)	−0.2300i	0.2403	24.5837 + 24.6586i	24.5837 − 24.6586i	1.7827	1.7831
	(2, 1)	−0.6074i	1.0633	24.1252 + 24.2048i	24.1252 − 24.2048i	3.6729	3.6733
	(1, 2)	−0.6570i	0.7126	24.4536 + 24.7538i	24.4536 − 24.7538i	0.8002	0.8003
	(2, 2)	−0.9526i	1.1544	24.3566 + 24.6606i	24.3566 − 24.6606i	1.8683	1.8685
0.001	(1, 1)	0.5010i	0.6527	54.8075 + 54.8414i	54.8075 − 54.8414i	1.7821	1.7823
	(2, 1)	−0.6074i	1.0632	54.0172 + 54.0528i	54.0172 − 54.0528i	3.6722	3.6724
	(1, 2)	−0.6566i	0.7125	54.9469 + 55.0812i	54.9469 − 55.0812i	0.7963	0.7964
	(2, 2)	0.9522i	1.1540	54.7338 + 54.8697i	54.7338 − 54.8697i	1.8654	1.8655

Table 8
Comparison of the frequency parameters $\Omega = \pi^2\omega\sqrt{\rho/G_{xy}}$ for closed circular cylindrical shells with F–F boundary conditions (E_x/E_y = 2, a/R = 10).

h/R	(m, n)	$\bar{\alpha}_1\tilde{a}/\pi$	$\bar{\alpha}_2\tilde{a}/\pi$	$\bar{\alpha}_3\tilde{a}/\pi$	$\bar{\alpha}_4\tilde{a}/\pi$	Exact [present]	S-DQFEM [app.]
0.05	(1, 1)	0.5480i	1.8143	7.0788 + 7.3831i	7.0788 − 7.3831i	2.3944	2.3945
	(2, 1)	0.6169i	1.2917	7.4128 + 7.6756i	7.4128 − 7.6756i	4.7299	4.7303
	(1, 2)	0.7269i	0.7741	7.3264 + 8.2723i	7.3264 − 8.2723i	1.2062	1.2113
	(2, 2)	−1.0327i	1.2817	7.2752 + 8.2381i	7.2752 − 8.2381i	2.3798	2.3831
0.005	(1, 1)	0.5510i	0.7870	24.3950 + 24.4718i	24.3950 − 24.4718i	2.3866	2.3864
	(2, 1)	−0.6170i	1.2918	28.8090 + 23.8921i	28.8090 − 23.8921i	4.7223	4.7220
	(1, 2)	−0.6976i	0.7655	24.4471 + 24.7476i	24.4471 − 24.7476i	0.9117	0.9116
	(2, 2)	−1.1076i	1.2773	24.3096 + 24.6154i	24.3096 − 24.6154i	2.2049	2.2048
0.001	(1, 1)	−0.5510i	0.7870	54.6175 + 54.6518i	54.6175 − 54.6518i	2.3866	2.3864
	(2, 1)	−0.6170i	1.2918	53.3127 + 53.3499i	53.3127 − 53.3499i	4.7222	4.7220
	(1, 2)	−0.6973i	0.7654	54.9327 + 55.0671i	54.9327 − 55.0671i	0.9083	0.9083
	(2, 2)	1.0174i	1.2773	54.6300 + 54.7667i	54.6300 − 54.7667i	2.2032	2.2031

Table 9
Comparison of the frequency parameters $\Omega = \pi^2\omega\sqrt{\rho/G_{xy}}$ for closed circular cylindrical shells with C–C boundary conditions (a/R = 5, h/R = 0.01).

E_x/E_y	(m, n)	$\bar{\alpha}_1\tilde{a}/\pi$	$\bar{\alpha}_2\tilde{a}/\pi$	$\bar{\alpha}_3\tilde{a}/\pi$	$\bar{\alpha}_4\tilde{a}/\pi$	Exact [present]	S-DQFEM [app.]
2	(1, 1)	−0.3072i	0.5740	8.4772 + 8.5343i	8.4772 − 8.5343i	4.0663	4.0665
	(2, 1)	0.2486i	1.0051	7.8862 + 7.9593i	7.8862 − 7.9593i	7.6458	7.6461
	(1, 2)	0.5138i	0.6484	8.5384 + 8.7549i	8.5384 − 8.7549i	2.2641	2.2642
	(2, 2)	−0.6519i	1.0602	8.33050 + 8.5643i	8.33050 − 8.5643i	4.6121	4.6123
5	(1, 1)	0.2099i	0.5370	6.7763 + 6.8356i	6.7763 − 6.8356i	4.1944	4.1946
	(2, 1)	0.1536i	1.0018	6.2961 + 6.3767i	6.2961 − 6.3767i	7.7992	7.7995
	(1, 2)	0.3885i	0.5971	6.8030 + 7.0253i	6.8030 − 7.0253i	2.5249	2.5250
	(2, 2)	0.4493i	1.0233	6.6190 + 6.8660i	6.6190 − 6.8660i	4.9142	4.9144
10	(1, 1)	−0.1536i	0.5207	5.7016 + 5.7673i	5.7016 − 5.7673i	4.2387	4.2389
	(2, 1)	−0.1076i	1.0020	5.2917 + 5.3832i	5.2917 − 5.3832i	7.8551	7.8554
	(1, 2)	0.3017i	0.5643	5.7007 + 5.9457i	5.7007 − 5.9457i	2.6322	2.6323
	(2, 2)	0.3294i	1.0121	5.5345 + 5.8107i	5.5345 − 5.8107i	5.0597	5.0599

Table 10
Comparison of the frequency parameters $\Omega = \pi^2\omega\sqrt{\rho/G_{xy}}$ for closed circular cylindrical shells with F–F boundary conditions (a/R = 5, h/R = 0.01).

E_x/E_y	(m, n)	$\bar{\alpha}_1\tilde{a}/\pi$	$\bar{\alpha}_2\tilde{a}/\pi$	$\bar{\alpha}_3\tilde{a}/\pi$	$\bar{\alpha}_4\tilde{a}/\pi$	Exact [present]	S-DQFEM [app.]
2	(1, 1)	−0.2860i	0.8520	8.1326 + 8.1982i	8.1326 − 8.1982i	6.4876	6.4873
	(2, 1)	0.1857i	1.1773	7.5785 + 7.6628i	7.5785 − 7.6628i	8.7662	8.7660
	(1, 2)	−0.5776i	0.7974	8.4812 + 8.7022i	8.4812 − 8.7022i	3.1135	3.1135
	(2, 2)	0.6923i	1.3216	8.1273 + 8.3801i	8.1273 − 8.3801i	5.9932	5.9931
5	(1, 1)	−0.1663i	0.9358	6.3781 + 6.4544i	6.3781 − 6.4544i	7.3562	7.3559
	(2, 1)	−0.1103i	1.1826	6.0563 + 6.1513i	6.0563 − 6.1513i	8.8870	8.8869
	(1, 2)	−0.4354i	0.8582	6.7030 + 6.9383i	6.7030 − 6.9383i	4.0269	4.0267
	(2, 2)	−0.4595i	1.3701	6.4026 + 6.6832i	6.4026 − 6.6832i	6.5731	6.5729
10	(1, 1)	0.1124i	0.9679	5.3272 + 5.4160i	5.3272 − 5.4160i	7.6354	7.6350
	(2, 1)	0.0764i	1.1840	5.0918 + 5.2003i	5.0918 − 5.2003i	8.9222	8.9221
	(1, 2)	−0.3272i	0.9076	5.5812 + 5.8482i	5.5812 − 5.8482i	4.5300	4.5298
	(2, 2)	0.3292i	1.3927	5.3327 + 5.6533i	5.3327 − 5.6533i	6.7885	6.7884

long circular cylindrical shells with E_x/E_y = 2 and C–C and F–F boundary conditions

$$\bar{\alpha}_3\sqrt{h} \approx 1.07 \tag{46}$$

In order to discuss the influences of the stiffness ratio E_x/E_y on the eigenvalues, exact frequency parameters are computed for the orthotropic circular cylindrical shells with C–C and F–F boundary conditions, a/R = 5, h/R = 0.01, and E_x/E_y = 2, 5, 10, as listed in Tables 9 and 10. One can see that the influences of the stiffness ratios E_x/E_y on the eigenvalues are apparent. For the sake of finding a simple relation to express the influences, the real part of $\bar{\alpha}_3$ for the fundamental frequencies is calculated using a/R = 10, E_x/E_y = 1, 2, 5, 10, 15, h/R = 0.1, 0.01, 0.005, 0.002, 0.001, and C–C and F–F boundary conditions, as shown in Fig. 3. It follows that, except for the cases of h/R = 0.1 (where a thin shell theory is questionable), all other lines are almost overlapped. Thus, the following equation is true for *thin* circular cylindrical shells with C–C and F–F boundary conditions

$$\bar{\alpha}_3\sqrt{h} \approx \sqrt[4]{\frac{2E_y}{E_x}} + 0.07 \tag{47}$$

Note that relations (46) and (47) are the same when E_x/E_y = 2, then it follows from Eq. (47) that the influences of quantities except E_x/E_y on the larger eigenvalues are small. One can also see from Tables 3–6 that boundary conditions have little effect on the two larger eigenvalues. Thus relation (47) can also be applied to free vibrations of circular cylindrical shells with other boundary conditions.

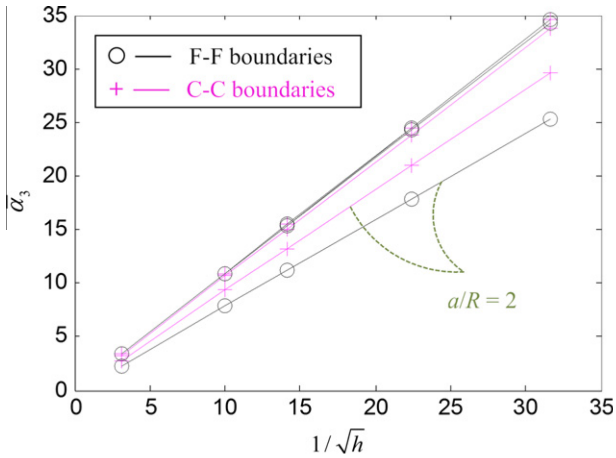

Fig. 2. The influences of length–radius ratio a/R and thickness–radius ratio h/R over the eigenvalues.

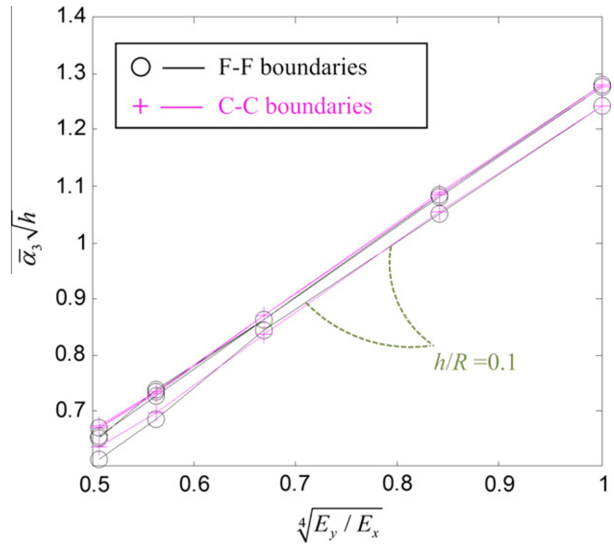

Fig. 3. The influences of stiffness ratio E_y/E_x and thickness–radius ratio h/R over the eigenvalues.

The two larger eigenvalues will induce computational problem for clamped-free shells, and the computational problem cannot be eliminated through formulation manipulations. The method to cope with the problem has been found, but it will not be presented here with consideration for the length of this paper.

5. Conclusion

This paper presented an exact procedure and closed-form solutions with analytically determined coefficients for free vibrations of thin orthotropic circular cylindrical shells with classical boundary conditions. The Donnell–Mushtari thin shell theory and the separation of variables method are employed in the derivation. The proposed method has been verified through comparing its results against results available in literature and of the highly accurate semi-analytical differential quadrature finite element method (S-DQFEM) developed by the authors. The characteristics of the eigenvalues are also examined.

It may be known that the Donnell–Mushtari shell theory is the simplest thin shell theory and its results for the lowest frequencies of a closed cylinder may not be as accurate [35,36]. This paper serves as a first step to get more accurate results for more accurate shell theories.

Acknowledgements

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Appendix A. The semi-analytical differential quadrature finite element method (S-DQFEM) for free vibrations of thin orthotropic circular cylindrical shells

The differential quadrature finite element method (DQFEM) is a combination of the differential quadrature method (DQM) and standard finite element method (FEM) [37,38]. DQFEM incorporates the high accuracy and efficiency of DQM and the simplicity of imposing boundary conditions of FEM. In this paper, semi-analytical formulation of the DQFEM is derived to calculate natural frequencies of free vibrating thin orthotropic circular cylindrical shells, which is superior to DQFEM in both simplicity and accuracy.

The potential energy and kinetic energy for thin orthotropic circular cylindrical shells are needed to derive the DQFEM formulations. They are

$$U=\frac{1}{2}\int_0^a\int_0^b\left[A_{11}\left(\frac{\partial u}{\partial\alpha}\right)^2+2A_{12}\frac{\partial u}{\partial\alpha}\left(\frac{\partial v}{\partial\beta}+\frac{w}{R}\right)+A_{22}\left(\frac{\partial v}{\partial\beta}+\frac{w}{R}\right)^2+A_{66}\left(\frac{\partial u}{\partial\beta}+\frac{\partial v}{\partial\alpha}\right)^2\right]\mathrm{d}\,\alpha\mathrm{d}\beta+\frac{k}{2}\int_0^a\int_0^b\left[A_{11}\left(\frac{\partial^2 w}{\partial\alpha^2}\right)^2+2A_{12}\frac{\partial^2 w}{\partial\alpha^2}\frac{\partial^2 w}{\partial\beta^2}+A_{22}\left(\frac{\partial^2 w}{\partial\beta^2}\right)^2+4A_{66}\left(\frac{\partial^2 w}{\partial\alpha\partial\beta}\right)^2\right]\mathrm{d}\,\alpha\mathrm{d}\beta \quad (A1)$$

$$T=\frac{1}{2}\omega^2\rho h\int_0^a\int_0^b[u^2+v^2+w^2]\mathrm{d}\,\alpha\,\mathrm{d}\,\beta \quad (A2)$$

For closed circular cylindrical shells, the modal displacements can be assumed as

$$\{u(\alpha,\beta),v(\alpha,\beta),w(\alpha,\beta)\}^{\mathrm{T}}=\{U(\alpha)\cos\Lambda\beta,V(\alpha)\sin\Lambda\beta,W(\alpha)\cos\Lambda\beta\}^{\mathrm{T}} \quad (A3)$$

where

$$\Lambda=\frac{n}{R} \quad (A4)$$

The substitution of expression (A3) into expressions (A1) and (A2) yields

$$U=\frac{1}{2}\frac{\Lambda b}{2}\int_0^a\left[A_{11}\left(\frac{\partial U}{\partial\alpha}\right)^2+2A_{12}\frac{\partial U}{\partial\alpha}\left(\Lambda V+\frac{W}{R}\right)+A_{22}\left(\Lambda V+\frac{W}{R}\right)^2+A_{66}\left(-\Lambda U+\frac{\partial V}{\partial\alpha}\right)^2\right]\mathrm{d}\alpha+\frac{k}{2}\frac{\Lambda b}{2}\int_0^a\left[A_{11}\left(\frac{\partial^2 W}{\partial\alpha^2}\right)^2-2A_{12}\Lambda^2W\frac{\partial^2 W}{\partial\alpha^2}+A_{22}\Lambda^4W^2+4A_{66}\Lambda^2\left(\frac{\partial W}{\partial\alpha}\right)^2\right]\mathrm{d}\alpha \quad (A5)$$

$$T=\frac{1}{2}\frac{\Lambda b}{2}\omega^2\rho h\int_0^a(U^2+V^2+W^2)\mathrm{d}\alpha \quad (A6)$$

which becomes one-dimensional problems.

Now we will use DQFEM to solve the problem. The nth derivatives of the field variable $f(x)$ at point x_i is approximated by DQM as [39]

$$f_i^{(n)}=\sum_{j=1}^{N}A_{ij}^{(n)}f_j\quad(i=1,2,\ldots,N) \quad (A7)$$

where $A_{ij}^{(n)}$ are the weighting coefficients of the nth order derivatives, and N the number of grid points in the x-direction. The

Gauss–Lobatto quadrature rule [37,38] with precision degree $(2N-3)$ for function $f(x)$ defined at $[-1, 1]$ is

$$\int_{-1}^{1} f(x)\mathrm{d}\,x = \sum_{j=1}^{N} C_j f(x_j) \tag{A8}$$

where C_j are the Gauss–Lobatto integration weights. The substitution of expressions (A7) and (A8) into expressions (A5) and (A6) yields

$$\begin{aligned} U = \frac{1}{2}\frac{\Lambda b}{2}\Big[&A_{11}\overline{\mathbf{U}}^{\mathrm{T}}\mathbf{A}^{(1)\mathrm{T}}\mathbf{C}bfA^{(1)}\overline{\mathbf{U}} + 2A_{12}\Lambda\overline{\mathbf{U}}^{\mathrm{T}}\mathbf{A}^{(1)\mathrm{T}}\mathbf{C}\overline{\mathbf{V}} + A_{22}\Lambda^2\overline{\mathbf{V}}^{\mathrm{T}}\mathbf{C}\overline{\mathbf{V}} \\ &+2A_{12}\overline{\mathbf{U}}^{\mathrm{T}}\mathbf{A}^{(1)\mathrm{T}}\mathbf{C}\overline{\mathbf{W}}/R + 2A_{22}\Lambda\overline{\mathbf{V}}^{\mathrm{T}}\mathbf{C}\overline{W}/R + A_{22}\overline{\mathbf{W}}^{\mathbf{T}}\mathbf{C}\overline{W}/R^2 \\ &+A_{66}\left(\Lambda^2\overline{\mathbf{U}}^{\mathrm{T}}\mathbf{C}\overline{\mathbf{U}} - 2\Lambda\overline{\mathbf{U}}^{\mathrm{T}}\mathbf{C}\mathbf{A}^{(1)}\overline{\mathbf{V}} + \overline{\mathbf{V}}^{\mathrm{T}}\mathbf{A}^{(1)\mathrm{T}}\mathbf{C}\mathbf{A}^{(1)}\overline{\mathbf{V}}\right)\Big] \\ &+\frac{k}{2}\frac{\Lambda b}{2}\overline{\mathbf{W}}^{\mathrm{T}}\Big[A_{11}\mathbf{A}^{(2)\mathrm{T}}\mathbf{C}\mathbf{A}^{(2)} - 2A_{12}\Lambda^2\mathbf{C}\mathbf{A}^{(2)} + A_{22}\Lambda^4\mathbf{C} \\ &+4A_{66}\Lambda^2\mathbf{A}^{(1)\mathrm{T}}\mathbf{C}\mathbf{A}^{(1)}\Big]\overline{\mathbf{W}} \end{aligned} \tag{A9}$$

$$T = \frac{1}{2}\frac{\Lambda b}{2}\omega^2\rho h\left(\overline{\mathbf{U}}^{\mathrm{T}}\mathbf{C}\overline{\mathbf{U}} + \overline{\mathbf{V}}^{\mathrm{T}}\mathbf{C}\overline{\mathbf{V}} + \overline{\mathbf{W}}^{\mathrm{T}}\mathbf{C}\overline{\mathbf{W}}\right) \tag{A10}$$

where

$$\overline{\mathbf{W}}^{\mathrm{T}} = [W_1\ W_2\ \cdots\ W_M] \tag{A11}$$

The variables $\overline{\mathbf{U}}$ and $\overline{\mathbf{V}}$ are of similar form as $\overline{\mathbf{W}}$. In order to construct elements satisfying C^1 inter-element continuity requirements for $\overline{\mathbf{W}}$, the element displacement vector should be

$$\mathbf{W}^{\mathrm{T}} = [W_1\ W_1'\ W_3\ \cdots\ W_{M-2}\ W_M\ W_M'] \tag{A12}$$

Using DQ rules one can find the relation between $\mathbf{W}$ and $\overline{\mathbf{W}}$ as

$$\mathbf{W} = \mathbf{Q}\overline{\mathbf{W}} \tag{A13}$$

where $\mathbf{Q}$ is a transformation matrix. Substitute Eq. (A13) into expressions (A9) and (A10) and applying the variational principle, one obtains the governing eigenvalue equation

$$(\mathbf{K} - \omega^2\mathbf{M})\begin{Bmatrix}\overline{\mathbf{U}}\\ \overline{\mathbf{V}}\\ \mathbf{W}\end{Bmatrix} = \mathbf{0} \tag{A14}$$

where

$$\mathbf{K} = A_{66}\frac{\Lambda b}{2}\begin{bmatrix} a_1\mathbf{A}^{(1)\mathrm{T}}\mathbf{C}\mathbf{A}^{(1)} + \Lambda^2\mathbf{C} & a_{12}\Lambda\mathbf{A}^{(1)\mathrm{T}}\mathbf{C} - \Lambda\mathbf{C}\mathbf{A}^{(1)} & a_{12}\mathbf{A}^{(1)\mathrm{T}}\mathbf{C}\mathbf{Q}^{-1}/R \\ a_{12}\Lambda\mathbf{C}\mathbf{A}^{(1)} - \Lambda\mathbf{A}^{(1)\mathrm{T}}\mathbf{C} & a_2\Lambda^2\mathbf{C} + \mathbf{A}^{(1)\mathrm{T}}\mathbf{C}\mathbf{A}^{(1)} & a_2\Lambda\mathbf{C}\mathbf{Q}^{-1}/R \\ a_{12}\mathbf{Q}^{-\mathrm{T}}\mathbf{C}\mathbf{A}^{(1)}/R & a_2\Lambda\mathbf{Q}^{-\mathrm{T}}\mathbf{C}/R & a_2\mathbf{Q}^{-\mathrm{T}}\mathbf{C}\mathbf{Q}^{-1}/R^2 + k\overline{\mathbf{K}}_{33} \end{bmatrix} \tag{A15}$$

$$\overline{\mathbf{K}}_{33} = \mathbf{Q}^{-\mathrm{T}}\left[a_1\mathbf{A}^{(2)\mathrm{T}}\mathbf{C}\mathbf{A}^{(2)} - a_{12}\Lambda^2(\mathbf{A}^{(2)\mathrm{T}}\mathbf{C} + \mathbf{C}\mathbf{A}^{(2)}) + a_2\Lambda^4\mathbf{C} + 4\Lambda^2\mathbf{A}^{(1)\mathrm{T}}\mathbf{C}\mathbf{A}^{(1)}\right]\mathbf{Q}^{-1} \tag{A16}$$

$$\mathbf{M} = \frac{\Lambda b}{2}\rho h\begin{bmatrix}\mathbf{C} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{C} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{Q}^{-\mathrm{T}}\mathbf{C}\mathbf{Q}^{-1}\end{bmatrix} \tag{A17}$$

The natural frequencies and mode shapes for free vibrations of thin orthotropic circular cylindrical shells can be obtained by solving Eq. (A14). If one wants to know more about DQM or DQFEM, please refer to the recent works by Xing and Liu [37,38] and the review paper by Bert and Malik [39].

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