Acta Mechanica Solida Sinica, Vol. 24, No. 6, December, 2011 ISSN 0894-9166
Published by AMSS Press, Wuhan, China

EXACT SOLUTIONS FOR FREE IN-PLANE VIBRATIONS OF RECTANGULAR PLATES**

Bo Liu* **Yufeng Xing**

(*The Solid Mechanics Research Center, Beijing University of Aeronautics and Astronautics, Beijing* 100191, *China*)

Received 23 October 2010, revision received 4 July 2011

ABSTRACT All possible exact solutions are successfully obtained in terms of 10 sets of distinct eigensolutions for the free in-plane vibration of isotropic rectangular plates. The plates have simply supported condition at two opposite edges and any combination of classical boundary conditions at the other two edges. The exact solutions are validated through both mathematical proof and comparisons with the solutions of differential quadrature method. Some unusual phenomena are revealed in free in-plane vibrations of rectangular plates due to one of the eigenvalues being zero. This work constitutes an improved version of very recent corresponding work by the same authors [Int. J. Mech. Sci., 2009, 51: 246-255]. Both the solution forms and solving procedures in the previous work are substantially simplified. Some new results are also given, which are useful for validation purpose in future.

KEY WORDS rectangular plate, in-pane free vibration, frequency, mode shape, exact solution

I. INTRODUCTION

There is an ongoing interest in the in-plane vibration due to a realization of its importance in recent years. The in-plane natural frequencies of plates tend to be very much higher than the lowest transverse natural frequencies, but some studies have shown that in-plane vibrations can play a major role for the transmissions of high-frequency vibration energies in structures[1–4], and any credible analysis technique must take into account the presence of these vibrations. The in-plane vibrations can also significantly affect the low frequency vibrations when two plates are connected at an angle along an edge[5]. In the case of plates subjected to tangential fluid boundary flow, such as on the hulls of ocean-going ships, in-plane vibration can be excited[6]. The in-plane vibration modes can be of very practical significance in aerospace structures[7], piezoelectric crystals and spur gears[8]. Thus the publications pertaining to in-plane vibrations are increasing in recent years.

A significant contribution to the topic of in-plane vibrations of rectangular plates was made by Bardell et al.[4]. They provided a number of in-plane frequencies for simply supported, clamped, and free plates, and referred to the pioneering work of Lord Rayleigh[9] dealing with what was referred to as 'simply supported' plates. And they have presented a valuable survey of much of the related literature available up to that time. The first known analytical study of free in-plane vibrations of non-simply supported rectangular plates was carried out by Gorman[6] who exactly analyzed plates with one pair of opposite edges simply supported, and another pair free or clamped. In Gorman's elegant work, only one quarter of the rectangular plate was analyzed, and it was shown that by this approach, the interpretation

* Corresponding author. E-mail: liubo68@buaa.edu.cn
** Project supported by the China Postdoctoral Science Foundation (No. 20100470179).

of the computed mode shapes with mode family separation became a much more manageable task, the probability of missing an eigenvalue could be greatly reduced, and the problem of repeated eigenvalues could be avoided. Two distinct classes of simply supported boundaries (denoted by the symbols S1 and S2 in this work) were clearly defined, which had a counterpart in the well known simply supported conditions utilized in the analysis of free lateral vibrations of rectangular plates. The work of Xing and Liu[10] was another significant contribution to analytically solving free in-plane vibrations of rectangular plates. They used the direct separation of variables method[11,12] to obtain exact solutions of the natural frequencies and the mode shapes for free in-plane vibrations of the rectangular plates, where at least two opposite edges had either type of simply supported conditions. All possible exact solutions were obtained, including the solutions for the cases S1-C-S2-F and S2-S1-S2-F etc. which were not available before. The entire plate was analyzed directly, and there were no problems in the interpretation of the computed mode shapes. The probabilities of missing an eigenvalue and the repeated eigenvalues were eliminated.

The free in-plane vibrations were also analyzed by researchers using approximate analytical approaches as well as numerical procedures. The superposition method was used by Gorman to study free in-plane vibrations of clamped and simply supported[13], completely free[14], and elastically supported[15] rectangular plates. Du et al.[5] presented an analytical method for the in-plane vibration analysis of rectangular plates with elastically restrained edges using an improved Fourier series method. Wang and Wereley[16], and Singh and Muhammad[7] analyzed free in-plane vibrations of rectangular and/or non-rectangular plates using the Kantorovich variational method and a modified form of the Rayleigh-Ritz method, respectively. Although Xing and Liu[10] have found exact solutions for free in-plane vibrations of rectangular plates, some inadequacy were found in the work when we tried to find exact solutions for free in-plane vibration of orthotropic rectangular plates[17]:

(1) The formulations of the exact solutions can be expressed in much simpler and more concise forms.

(2) The eigenvalues can be solved once by carrying all imaginary quantities through the solution process as imaginary instead of for three different cases.

(3) Mathematical proof of the exactness of the solution was not presented.

(4) The relations of the exact solutions for different boundary conditions are not revealed clearly.

(5) Some unusual phenomena in free in-plane vibrations of rectangular plates due to one of the eigenvalues being zero were not addressed clearly.

These deficiencies motivated us to address this work. Fortunately, all deficiencies are successfully solved, and the techniques developed in this work and some results of this work are guide lines for obtaining exact solutions for free in-plane vibration of orthotropic rectangular plates[17].

If possible and practical, the equations of free in-plane vibrations should be solved exactly because it is difficult to assess the accuracy of an approximate solution. Moreover, some phenomena, for example, some frequencies and/or mode shapes for free in-plane vibration of rectangular plates with different boundary conditions and aspect ratio may be the same, which cannot be easily explained if a numerical method is used but they can be easily explained if an exact method is used. It is well known that, of all available solutions, those based on an exact approach, wherein the governing equations and the boundary conditions are satisfied rigorously, are valuable and computationally efficient[18].

The organization of this paper is as follows. In §II the governing differential equations and boundary conditions are presented. In §III, the problems in Ref.[10] are described and improvements are presented. In §IV, some new exact results are presented. Finally, conclusions are outlined in §V.

II. DIFFERENTIAL EQUATIONS AND BOUNDARY CONDITIONS

The differential equations and boundary conditions for free in-plane vibrations of a rectangular plate (Fig.1) can be found in literature [10]. They are presented here for the derivation of exact solutions. The stress components for the plane stress problems can be expressed in terms of displacements as

$$\sigma_x = C\left(\frac{\partial u}{\partial x} + \nu\frac{\partial v}{\partial y}\right), \quad \sigma_y = C\left(\frac{\partial v}{\partial y} + \nu\frac{\partial u}{\partial x}\right), \quad \tau_{xy} = G\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \tag{1}$$

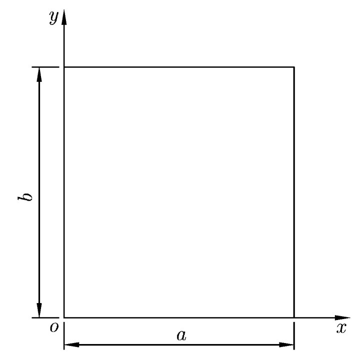


Fig. 1. A schematic of rectangular plate and its coordinates.

where $C = E/(1-\nu^2)$, E is the Young's modulus, $G = E/[2(1+\nu)]$ the shear modulus, ν the Poisson's ratio. The governing differential equations for the free in-plane vibration are

$$\frac{\partial^2 u}{\partial x^2} + \nu_1 \frac{\partial^2 u}{\partial y^2} + \nu_2 \frac{\partial^2 v}{\partial x \partial y} + \nu_1 \beta^2 u = 0 \tag{2a}$$

$$\frac{\partial^2 v}{\partial y^2} + \nu_1 \frac{\partial^2 v}{\partial x^2} + \nu_2 \frac{\partial^2 u}{\partial x \partial y} + \nu_1 \beta^2 v = 0 \tag{2b}$$

where

$$\nu_1 = \frac{1-\nu}{2},\ \nu_2 = \frac{1+\nu}{2},\ \beta = \omega\sqrt{\frac{\rho}{G}} \tag{3}$$

and ρ is the volume density. The boundary conditions include clamped edge, free edge and simply supported edge. We seek separation of variables solutions as

$$u(x,y) = u_1(x)\, u_2(y)\,, \quad v(x,y) = v_1(x)\, v_2(y) \tag{4}$$

then the boundary conditions used to solve exact solutions for free in-plane vibrations can be specified as

(1) Simply supported edge, where the normal stress and tangential displacement are zero (S1) or normal displacement and tangential stress are zero (S2). For $x = a$, those two kinds of simply supported conditions are

$$\text{S1}: \quad \sigma_x = C\left(\frac{\partial u}{\partial x} + \nu \frac{\partial v}{\partial y}\right) = 0,\ v = 0 \quad \Rightarrow \quad u_1'(a) = 0, \quad v_1(a) = 0 \tag{5}$$

$$\text{S2}: \quad \tau_{xy} = G\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = 0,\ u = 0 \quad \Rightarrow \quad u_1(a) = 0, \quad v_1'(a) = 0 \tag{6}$$

For $y = b$, the two kinds of simply supported conditions are

$$\text{S1}: \quad \sigma_y = C\left(\frac{\partial v}{\partial y} + \nu \frac{\partial u}{\partial x}\right) = 0,\ u = 0 \quad \Rightarrow \quad v_2'(b) = 0, \quad u_2(b) = 0 \tag{7}$$

$$\text{S2}: \quad \tau_{xy} = G\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = 0,\ v = 0 \quad \Rightarrow \quad v_2(b) = 0, \quad u_2'(b) = 0 \tag{8}$$

(2) Clamped edge (C), where displacements on the boundary are zero. For $y = b$, we have

$$u = 0,\ v = 0 \quad \Rightarrow \quad u_2(b) = 0, \quad v_2(b) = 0 \tag{9}$$

(3) Free edge (F), where stress components are zeros. For $y = b$, we have

$$\sigma_y = C\left(\frac{\partial v}{\partial y} + \nu \frac{\partial u}{\partial x}\right) = 0,\ \tau_{xy} = G\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) = 0 \Rightarrow \begin{cases} v_1(x)\, v_2'(b) + \nu u_1'(x)\, u_2(b) = 0 \\ v_1'(x)\, v_2(b) + u_1(x)\, u_2'(b) = 0 \end{cases} \tag{10}$$

There are two classes of simply supported edges for the free in-plane vibrations, which are different from the lateral vibrations of thin rectangular plates. Therefore, the combinations of above boundary conditions are much more complicated than the counterpart in lateral vibrations of thin rectangular plates.

III. EXACT EIGENSOLUTIONS

3.1. Separation of Variables Method

Substitute Eq.(2a) into Eq.(2b) to eliminate v yields

$$\nabla^4 u + (1+\nu_1)\beta^2\nabla^2 u + \nu_1\beta^4 u = 0 \tag{11}$$

where ∇^2 is the Laplace operator. According to the direct separation of variables method[10,11], the characteristic equation of Eq.(11) can be obtained by substituting $u = \mathrm{e}^{\mu x}\mathrm{e}^{\lambda y}$ into Eq.(11) as

$$\left(\mu^2+\lambda^2\right)^2 + (1+\nu_1)\left(\mu^2+\lambda^2\right)\beta^2 + \nu_1\beta^4 = 0 \tag{12}$$

Solving Eq.(12), we have

$$\mu^2+\lambda^2 = -\beta^2 \quad \text{or} \quad \mu^2+\lambda^2 = -\nu_1\beta^2 \tag{13}$$

We can further obtain

$$\mu_{1,2} = \pm\mathrm{i}\Omega,\ \mu_{3,4} = \pm\mathrm{i}\Lambda \quad \text{or} \quad \lambda_{1,2} = \pm\mathrm{i}T,\ \lambda_{3,4} = \pm\mathrm{i}Z \tag{14}$$

where

$$\Omega = \sqrt{\lambda^2+\beta^2},\ \Lambda = \sqrt{\lambda^2+\nu_1\beta^2} \tag{15}$$

$$T = \sqrt{\mu^2+\beta^2},\ Z = \sqrt{\mu^2+\nu_1\beta^2} \tag{16}$$

Xing and Liu[10] assumed the exact solutions as

$$u(x,y) = \phi_1(x)\phi_2(y), \quad v(x,y) = \psi_1(x)\psi_2(y) \tag{17}$$

where

$$\begin{aligned}
\phi_1(x) &= A_1\cos(\Omega x) + A_2\sin(\Omega x) + A_3\cos(\Lambda x) + A_4\sin(\Lambda x)\\
\psi_1(x) &= C_1\cos(\Omega x) + C_2\sin(\Omega x) + C_3\cos(\Lambda x) + C_4\sin(\Lambda x)\\
\phi_2(y) &= B_1\cos(Ty) + B_2\sin(Ty) + B_3\cos(Zy) + B_4\sin(Zy)\\
\psi_2(y) &= D_1\cos(Ty) + D_2\sin(Ty) + D_3\cos(Zy) + D_4\sin(Zy)
\end{aligned} \tag{18}$$

They found that one pair of opposite edges have to be simply supported for obtaining exact eigensolutions in separation of variables form, for example, $x = 0$ and a. The substitution of $\mu = \mathrm{i}\Omega$ into Eq.(16) yields

$$T = \sqrt{\beta^2-\Omega^2},\ Z = \sqrt{\nu_1\beta^2-\Omega^2} \tag{19}$$

According to $\Omega^2 \le \nu_1\beta^2$, $\nu_1\beta^2 < \Omega^2 \le \beta^2$, or $\beta^2 < \Omega^2$, Xing and Liu[10] presented different formulations for $\phi_2(y)$ and $\psi_2(y)$, and obtained the corresponding relations for B_j and D_j, where $j = 1$, 2, 3, 4. Substituting Eq.(17) into boundary conditions, they presented the expressions of D_j for three cases. In fact, the exact solutions can be given in a compact and elegant form as follows:

$$u(x,y) = u_1(x)\,u_2(y) = k_1\phi_1'\,(\psi_1+\psi_2) \tag{20a}$$

$$v(x,y) = v_1(x)\,v_2(y) = \phi_1\,(k_3\psi_1' + k_4\psi_2') \tag{20b}$$

where

$$\phi_1 = A_1\cos(\Omega x) + A_2\sin(\Omega x) \tag{21}$$

$$\psi_1 = B_1\cos(Ty) + B_2\sin(Ty) \tag{22a}$$

$$\psi_2 = B_3\cos(Zy) + B_4\sin(Zy) \tag{22b}$$

The coefficients k_1, k_3 and k_4 will be determined below according to governing differential equations, while A_1, A_2, B_1, …, B_4 are determined by substituting Eqs.(20) into boundary conditions. It is noteworthy that eigenvalues T and Z in Eqs.(22) can be real or imaginary, which is different from Ref.[10].

Table 1. **The eigenequations and eigenfunctions of x coordinates with two types simple support edges**

Boundary conditions	Eigenequations	Eigenfunctions
S2-S2	$\sin(\Omega a)=0$	$u_1(x)=k_1\sin(\Omega x),\quad v_1(x)=\cos(\Omega x)$
S1-S1	$\sin(\Omega a)=0$	$u_1(x)=k_1\cos(\Omega x),\quad v_1(x)=\sin(\Omega x)$
S2-S1	$\cos(\Omega a)=0$	$u_1(x)=k_1\sin(\Omega x),\quad v_1(x)=\cos(\Omega x)$
S1-S2	$\cos(\Omega a)=0$	$u_1(x)=k_1\cos(\Omega x),\quad v_1(x)=\sin(\Omega x)$

3.2. The Eigensolutions for Simply Supported Plates

Assume the pair of opposite edges in x coordinates to be simply supported. The substitution of Eqs.(20) into the simply supported boundary conditions, namely Eqs.(5) and (6), yields the eigenequations and eigenfunctions of x coordinates, as listed in Table 1. Note that Ω in Table 1 can also be Λ, that is, they can be taken as the same for y coordinates. In Xing and Liu[10], k_1 in Table 1 was not used because it could be included in the factor of the eigenfunctions of y coordinates. But we can see below that the use of k_1 is necessary because Ω or T (or Z) may become zero.

Similarly, assume the pair of opposite edges in y coordinates to be simply supported, we can also get the corresponding eigenequations and eigenfunctions of y coordinates as listed in Table 2. Since T and Z correspond to different frequency parameters β, the eigenfunctions are different, although the eigenequations are the same for a given boundary condition. One can see from Tables 1 and 2 that two kinds of eigenfuctions are derived in all four kinds of combinations of S1 and S2 for each coordinate. The two kinds of simply supported conditions for the four edges have six distinct combinations, of which three kinds of different characteristic functions and equations are obtained as shown in Table 3.

Table 2. **The eigenequations and eigenfunctions of y coordinates with two types simple support edges**

Boundary conditions	Eigenequations	Eigenfunctions
S2-S2	$\sin(Tb)=0$	$u_2(y)=\cos(Ty),\quad v_2(y)=k_3\sin(Ty)$
	or $\sin(Zb)=0$	or $u_2(y)=\cos(Zy),\quad v_2(y)=k_4\sin(Zy)$
S1-S1	$\sin(Tb)=0$	$u_2(y)=\sin(Ty),\quad v_2(y)=k_3\cos(Ty)$
	or $\sin(Zb)=0$	or $u_2(y)=\sin(Zy),\quad v_2(y)=k_4\cos(Zy)$
S2-S1	$\cos(Tb)=0$	$u_2(y)=\cos(Ty),\quad v_2(y)=k_3\sin(Ty)$
	or $\cos(Zb)=0$	or $u_2(y)=\cos(Zy),\quad v_2(y)=k_4\sin(Zy)$
S1-S2	$\cos(Tb)=0$	$u_2(y)=\sin(Ty),\quad v_2(y)=k_3\cos(Ty)$
	or $\cos(Zb)=0$	or $u_2(y)=\sin(Zy),\quad v_2(y)=k_4\cos(Zy)$

Table 3. **Characteristic functions and equations for six different combinations of simple support edges**

Boundary conditions	Characteristic functions	Characteristic equations
S1-S1-S1-S1	$u(x,y)=T\cos(\Omega x)\sin(Ty),\ v(x,y)=-\Omega\sin(\Omega x)\cos(Ty)$	$\beta=\sqrt{\Omega^2+T^2}$
	$u(x,y)=\Omega\cos(\Omega x)\sin(Zy),\ v(x,y)=Z\sin(\Omega x)\cos(Zy)$	$\beta=\sqrt{(\Omega^2+Z^2)/\nu_1}$
S2-S2-S2-S2	$u(x,y)=T\sin(\Omega x)\cos(Ty),\ v(x,y)=-\Omega\cos(\Omega x)\sin(Ty)$	$\beta=\sqrt{\Omega^2+T^2}$
	$u(x,y)=\Omega\sin(\Omega x)\cos(Zy),\ v(x,y)=Z\cos(\Omega x)\sin(Zy)$	$\beta=\sqrt{(\Omega^2+Z^2)/\nu_1}$
S1-S2-S1-S1 S1-S2-S2-S1 S1-S2-S1-S2 S1-S2-S2-S2	$u(x,y)=T\cos(\Omega x)\cos(Ty),\ v(x,y)=\Omega\sin(\Omega x)\sin(Ty)$	$\beta=\sqrt{\Omega^2+T^2}$
	$u(x,y)=\Omega\cos(\Omega x)\cos(Zy),\ v(x,y)=-Z\sin(\Omega x)\sin(Zy)$	$\beta=\sqrt{(\Omega^2+Z^2)/\nu_1}$

One can also see from Table 3 that, for the first kind of characteristic functions of S1S1S1S1 plates, one of the two eigenvalues T and Ω can be zero; however, for the other kind of characteristic functions, neither Ω nor Z can be zero. Similar phenomenon exists in the other five types of plates. The characteristic functions in Table 3 can also be used to judge whether Ω or T (or Z) can be zero for free in-plane vibrations of simply supported orthotropic rectangular plates[17] where they cannot be directly judged from the characteristic functions. It deserves to be noted that when one eigenvalue becomes zero, the corresponding frequencies will not change with the length of the corresponding side.

3.3. The Characteristic Solutions for Arbitrary Opposite Edges $y = 0$ and b

Assume the pair of opposite edges of x coordinates to be simply supported. The eigenfunctions of x coordinates are listed in Table 1. There are two kinds of characteristic functions for the four combinations of S1 and S2. The characteristic functions for one kind are

$$u(x,y) = k_1(\psi_1 + \psi_2)\sin(\Omega x), \quad v(x,y) = (k_3\psi_1' + k_4\psi_2')\cos(\Omega x) \tag{23}$$

We denote them as the first case. The substitution of Eq.(23) into the first governing differential equation yields

$$\begin{aligned}\frac{\partial^2 u}{\partial x^2} + \nu_1\frac{\partial^2 u}{\partial y^2} + \nu_2\frac{\partial^2 v}{\partial x\partial y} + \nu_1\beta^2 u &= \left(-k_1\Omega^2 - \nu_1 k_1 T^2 + \nu_2 k_3\Omega T^2 + \nu_1 k_1\beta^2\right)\psi_1\sin(\Omega x)\\ &+ \left(-k_1\Omega^2 - \nu_1 k_1 Z^2 + \nu_2 k_4\Omega Z^2 + \nu_1 k_1\beta^2\right)\psi_2\sin(\Omega x) = 0\end{aligned} \tag{24}$$

Using Eq.(19), one can obtain $k_1/k_3 = T^2/\Omega$ and $k_1/k_4 = -\Omega$ from Eq.(24). Because Ω and T may be zero, we take $k_1 = \Omega T^2$, then

$$k_3 = \Omega^2, \quad k_4 = -T^2 \tag{25}$$

The substitution of Eq.(23) into another governing differential equation leads to

$$\begin{aligned}\frac{\partial^2 v}{\partial y^2} + \nu_1\frac{\partial^2 v}{\partial x^2} + \nu_2\frac{\partial^2 u}{\partial x\partial y} + \nu_1\beta^2 v &= \left(-k_3 T^2 - \nu_1 k_3\Omega^2 + \nu_2 k_1\Omega + \nu_1 k_3\beta^2\right)\psi_1'\cos(\Omega x)\\ &+ \left(-k_4 Z^2 - \nu_1 k_4\Omega^2 + \nu_2 k_1\Omega + \nu_1 k_4\beta^2\right)\psi_2'\cos(\Omega x) = 0\end{aligned} \tag{26}$$

Using Eq.(19), we can derive the same results as Eq.(25). So expressions in Eq.(23) satisfy the governing differential equations exactly with the coefficients of Eq.(25). It may be seen that, however, exact solutions are only available for the rectangular plates with one pair of opposite edges simply supported due to the coupling of the eigenvalues of the two variables.

The characteristic functions of the other kind are

$$u(x,y) = \bar{k}_1(\psi_1 + \psi_2)\cos(\Omega x), \quad v(x,y) = \left(\bar{k}_3\psi_1' + \bar{k}_4\psi_2'\right)\sin(\Omega x) \tag{27}$$

which are denoted as the second case here. The substitution of Eq.(27) into the governing differential equations yields $\bar{k}_1 = \Omega T^2$ also, but

$$\bar{k}_3 = -\Omega^2, \quad \bar{k}_4 = T^2 \tag{28}$$

Note that the signs of Eq.(28) and Eq.(25) are opposite. Thus the two cases can be written in a unified form as

$$u(x,y) = k_1(\psi_1 + \psi_2)\phi_1(x), \quad v(x,y) = (k_3\psi_1' + k_4\psi_2')\phi_2(x) \tag{29}$$

where

$$\phi_1(x) = \sin(\Omega x), \quad \phi_2(x) = \cos(\Omega x) \tag{30}$$

for the first case and

$$\phi_1(x) = \cos(\Omega x), \quad \phi_2(x) = -\sin(\Omega x) \tag{31}$$

for the second case. Substituting Eq.(29) into the boundary conditions yields the characteristic equations and functions for in-plane vibrations of rectangular plates with one pair of simply supported opposite edges ($x = 0$, a) as listed below:

Case S1-C:

$$\begin{aligned}&TZ\sin(Tb)\cos(Zb) + \Omega^2\sin(Zb)\cos(Tb) = 0\\ &u(x,y) = \Omega T\phi_1(x)[\sin(Tb)\sin(Zy) - \sin(Zb)\sin(Ty)]\\ &v(x,y) = -\phi_2(x)[TZ\sin(Tb)\cos(Zy) + \Omega^2\sin(Zb)\cos(Ty)]\end{aligned} \tag{32}$$

Case S2-C:

$$\begin{aligned}&\Omega^2\cos(Zb)\sin(Tb) + TZ\cos(Tb)\sin(Zb) = 0\\ &u(x,y) = \Omega T\phi_1(x)[\cos(Tb)\cos(Zy) - \cos(Zb)\cos(Ty)]\\ &v(x,y) = \phi_2(x)[TZ\cos(Tb)\sin(Zy) + \Omega^2\cos(Zb)\sin(Ty)]\end{aligned} \tag{33}$$

Case C-C:

$$\begin{aligned}&(\Omega^4+T^2Z^2)\sin(Tb)\sin(Zb)+2\Omega^2TZ[1-\cos(Tb)\cos(Zb)]=0\\&u(x,y)=\Omega T\phi_1(x)\{\Omega^2\sin(Zy)+TZ\sin(Ty)-q_1[\cos(Ty)-\cos(Zy)]\}\\&v(x,y)=\phi_2(x)\{\Omega^2TZ[\cos(Ty)-\cos(Zy)]+q_1[\Omega^2\sin(Ty)+TZ\sin(Zy)]\}\end{aligned}\tag{34}$$

where

$$q_1=\frac{\Omega^2\sin(Zb)+TZ\sin(Tb)}{\cos(Tb)-\cos(Zb)}\tag{35}$$

Case S1-F:

$$\begin{aligned}&L_1L_4\cos(Tb)\sin(Zb)-L_2L_3\cos(Zb)\sin(Tb)=0\\&u(x,y)=\Omega\phi_1(x)[2TZ\cos(Zb)\sin(Ty)-L_1\cos(Tb)\sin(Zy)]\\&v(x,y)=Z\phi_2(x)[2\Omega^2\cos(Zb)\cos(Ty)+L_1\cos(Tb)\cos(Zy)]\end{aligned}\tag{36}$$

Case S2-F:

$$\begin{aligned}&L_1L_4\cos(Zb)\sin(Tb)-L_2L_3\cos(Tb)\sin(Zb)=0\\&u(x,y)=\ \Omega\phi_1(x)[2TZ\sin(Zb)\cos(Ty)-L_1\sin(Tb)\cos(Zy)]\\&v(x,y)=-Z\phi_2(x)[2\Omega^2\sin(Zb)\sin(Ty)+L_1\sin(Tb)\sin(Zy)]\end{aligned}\tag{37}$$

Case C-F:

$$\begin{aligned}&\Omega^2[L_5+L_6\sin(Tb)\sin(Zb)]+TZL_7\cos(Tb)\cos(Zb)=0\\&u(x,y)=\Omega\phi_1(x)\{\Omega^2\sin(Zy)+TZ\sin(Ty)+q_2TZ[\cos(Ty)-\cos(Zy)]\}\\&v(x,y)=Z\phi_2(x)\{\Omega^2[\cos(Ty)-\cos(Zy)]-q_2[TZ\sin(Zy)+\Omega^2\sin(Ty)]\}\end{aligned}\tag{38}$$

where

$$q_2=\frac{2\Omega^2\cos(Zb)+L_1\cos(Tb)}{2TZ\sin(Zb)-L_1\sin(Tb)}\tag{39}$$

Case F-F:

$$\begin{aligned}&2L_1L_2L_3L_4[1-\cos(Tb)\cos(Zb)]-(L_1^2L_4^2+L_2^2L_3^2)\sin(Tb)\sin(Zb)=0\\&u(x,y)=\Omega\phi_1(x)\{\ q_4[L_1\sin(Zy)\ -\ L_2\sin(Ty)]\ -\ L_1L_2q_3[L_4\cos(Ty)\ -\ L_3\cos(Zy)]\}\\&v(x,y)=Z\phi_2(x)\{q_3\Omega^2L_1[2L_4\sin(Ty)+(m-1)L_2\sin(Zy)]-q_4[L_1\cos(Zy)+2\Omega^2\cos(Ty)]\}\end{aligned}\tag{40}$$

where

$$q_3=\cos(Tb)-\cos(Zb),\quad q_4=L_1L_4\sin(Tb)-L_2L_3\sin(Zb)\tag{41}$$

The introduced parameters L_j $(j=1,\ldots,7)$ in Eqs.(36) to (41) are defined as follows:

$$L_1=T^2-\Omega^2,\ \ L_2=2TZ,\ \ L_3=(\nu-1)\Omega^2,\ \ L_4=\nu\Omega^2+Z^2\tag{42}$$

$$L_5=TZ[(m-1)L_1-2L_4],\quad L_6=L_1L_4-(m-1)TZL_2,\quad L_7=2L_3\Omega^2-L_1L_4\tag{43}$$

Comparing Eqs.(32)-Eq.(43) with the exact solutions of Xing and Liu[10], we can see that the characteristic equations and functions are more concise.

As in the case of simply supported plates, it deserves special attentions when the boundary conditions of the edges $x=0$ and a are S2-S2 or S1-S1 because Ω may equal zero. It follows that $k_1=0$, $k_3=0$, and $k_4\neq 0$ if $\Omega=0$, see Eqs.(25) or (28). The eigenequations and characteristic functions of plates with simply supported conditions at $x=0$, a and $\Omega=0$ have been derived and listed in Table 4. One may

Table 4. **The eigenequations and characteristic functions for plates with $x=0$, a simply supported and $\Omega=0$**

B.C.s of $x=0$, a	B.C.s of $y=0$, b	Eigenequations	Characteristic functions
S1-S1	S1-S1, S1-C, C-C	$\sin(Tb)=0$	$u(x,y)=\sin(Ty),\quad v(x,y)=0$
	S1-S2, S1-F, C-F	$\cos(Tb)=0$	$u(x,y)=\sin(Ty),\quad v(x,y)=0$
	S2-S1, S2-C	$\cos(Tb)=0$	$u(x,y)=\cos(Ty),\quad v(x,y)=0$
	S2-S2, S2-F, F-F	$\sin(Tb)=0$	$u(x,y)=\cos(Ty),\quad v(x,y)=0$
S2-S2	S1-S1, S1-F, F-F	$\sin(Zb)=0$	$u(x,y)=0,\quad v(x,y)=\cos(Zy)$
	S1-S2, S1-C	$\cos(Zb)=0$	$u(x,y)=0,\quad v(x,y)=\cos(Zy)$
	S2-S1, S2-F, C-F	$\cos(Zb)=0$	$u(x,y)=0,\quad v(x,y)=\sin(Zy)$
	S2-S2, S2-C, C-C	$\sin(Zb)=0$	$u(x,y)=0,\quad v(x,y)=\sin(Zy)$

have noted that the frequencies and/or mode shapes of free in-plane vibration of rectangular plates with different boundary conditions at $y = 0$, b may be the same as those when the edges at $x = 0$, a are simply supported and $\Omega = 0$. These phenomena are unusual when compared with the lateral free vibration of rectangular plates, and may have some potential applications or disadvantages in designing or analyzing piezoelectric crystal devices.

Xing and Liu[10] discussed the relationship of frequencies and mode shapes for the rectangular plates with one pair of simply supported opposite edges being S1-S1 or S2-S2, another pair being C-C, F-F and C-F; but the characteristic functions for the pair of opposite edges of x coordinates being S1-S1 when $\Omega = 0$ were not presented.

In a word, in this section, we simplified the forms of the exact eigensolutions greatly. All possible exact solutions are solved and expressed by 10 distinct eigensolutions and the mathematical proof of the exactness of the solution is presented. In addition, the three different cases used by Xing and Liu[10] and Gorman[6] to solve exact solutions are simplified and solved once, and the unusual phenomenon when $\Omega = 0$ is also discussed.

IV. NUMERICAL SOLUTIONS

In this section, some new exact results are presented, against which any approximate method for free in-plane vibrations can be validated, since the results in Ref.[10] are limited in the three cases C-C, C-F, and F-F for the y coordinates and the aspect ratio of $b/a = 1.2$. The frequency parameters are solved from the characteristic equations by using of Newton-Raphson method. Although numerical computation is required, the results are exact in the same sense as that the numerical solution to the transcendental

Table 5. **The dimensionless frequency parameters $\gamma = \beta a/\pi$ and eigenvalues for square plates with S1-S1 in x-coordinate**

B.C.		1	2	3	4	5	6	7	8	9	10
S1-S1	$\Omega a/\pi$	1	0	1	2	0	1	2	1	2	3
	Tb/π	0	1	1	0	2	2	1	(1)#	2	0
	γ	1	1	1.4142	2	2	2.2361	2.2361	2.3905	2.8284	3
	γ^*	1.0000	1.0000	1.4142	2.0000	2.0000	2.2361	2.2361	2.3905	2.8284	3.0000
S1-C	$\Omega a/\pi$	0	1	1	0	2	1	2	1	0	3
	Tb/π	1	0.6386	1.5459	2	0.5704	2.0511	1.6552	2.7561	3	0.5460
	Zb/π	1.8586	2.2375i	1.3565	3.7172	4.9535i	2.8491	4.0245i	4.4524	5.5758	7.5304i
	γ	1	1.1865	1.8412	2	2.0798	2.2819	2.5961	2.9319	3	3.0493
S2-C	$\Omega a/\pi$	0	0	1	1	2	1	0	1	2	3
	Tb/π	0.5	1.5	1.1344	1.5641	1.1246	2.1025	2.5	2.5726	2.1551	1.0874
	Zb/π	0.9293	2.7879	1.4036i	1.4268	4.6143i	2.9756	4.6465	4.0555	3.1013i	7.3248i
	γ	0.5	1.5	1.5122	1.8565	2.2945	2.3282	2.5	2.7601	2.9401	3.1910
C-C	$\Omega a/\pi$	0	1	1	0	2	1	0	2	1	3
	Tb/π	1	1.4510	1.5923	2	1.2773	2.4726	3	2.2687	2.8587	1.1892
	Zb/π	1.8586	0.9260	1.5306	3.7172	4.4750i	3.8347	5.5758	2.8073i	4.6706	7.2700i
	γ	1	1.7622	1.8803	2	2.3731	2.6672	3	3.0244	3.0286	3.2271
S1-F	$\Omega a/\pi$	0	1	1	0	2	1	2	1	0	3
	Tb/π	0.5	0.4861i	1	1.5	0.8195i	1.5633	0.8479	2.0896	2.5	1.2044i
	Zb/π	0.9293	2.6891i	2.7207i	2.7879	5.2897i	1.4236	4.8143i	2.9442	4.6465	7.9214i
	γ	0.5	0.8739	1.4142	1.5	1.8244	1.8558	2.1723	2.3166	2.5	2.7476
S2-F	$\Omega a/\pi$	0	1	1	2	1	0	1	2	3	1
	Tb/π	1	0.0911	1.3075	0.7795i	1.5479	2	2.0614	1.4659	1.1996i	2.7308
	Zb/π	1.8586	2.5272i	0.7139i	5.2688i	1.3644	3.7172	2.8747	4.2706i	7.9189i	4.3982
	γ	1	1.0041	1.6461	1.8418	1.8428	2	2.2912	2.4797	2.7497	2.9081
C-F	$\Omega a/\pi$	0	1	0	1	2	1	1	0	2	3
	Tb/π	0.5	0.3634	1.5	1.3090	0.7590i	2.0039	2.1820	2.5	1.6418	1.1973i
	Zb/π	0.9293	2.4411i	2.7879	0.7048i	5.2584i	2.7306	3.1672	4.6465	4.0434i	7.9176i
	γ	0.5	1.0640	1.5	1.6472	1.8504	2.2396	2.4002	2.5	2.5876	2.7507
F-F	$\Omega a/\pi$	1	0	1	2	1	1	0	2	1	3
	Tb/π	0.6531i	1	1	0.9721i	1.5086	1.5949	2	0.1822	2.5177	1.2879i
	Zb/π	2.8087i	1.8586	1.7207i	5.3782i	1.2025	1.5400	3.7172	5.0543i	3.9345	7.9666i
	γ	0.7573	1	1.4142	1.7479	1.8099	1.8825	2	2.0083	2.7090	2.7095

#The value in parentheses corresponds to Z and the second type characteristic function and equation of Table 3, γ^* denotes frequencies by DQM. $\mathrm{i} = \sqrt{-1}$

frequency equation for a beam yields an exact solution[20]. Different initial trial values usually converge to the nearest exact eigenvalues and have little effect on the accuracy of Newton-Raphson method.

Tables 5 and 6 include dimensionless frequency parameters and eigenvalues for free in-plane vibrations of the square plates with the pair of opposite edges of x coordinates being S1-S1 and S1-S2, respectively, and another pair of opposite edges being several combinations of S1, S2, C, and F. The formulations of the dimensionless frequency parameters and eigenvalues are included in the tables. The frequencies and eigenvalues of the plates with the pair of opposite edges of y coordinates being any combination of classical boundary conditions, while the pair of opposite edges of x coordinates being S2-S2 are the same as those of the plates with the pair of opposite edges of x coordinates being S1-S1 if $\Omega \neq 0$. The frequencies and eigenvalues for $\Omega = 0$ of these cases can be easily calculated (see Table 4). Thus, the frequency parameters for the pair of opposite edges of x coordinates being S2-S2 are not presented. All frequency parameters and mode shapes have been checked using the differential quadrature method (DQM). Literatures about DQM are abundant and one can read the review paper of Bert and Malik[21] and a recent publication of Xing and Liu[22] to make a quick understanding of DQM.

Table 6. **The dimensionless frequency parameters $\gamma = \beta a/\pi$ and eigenvalues for square plates with S1-S2 in x-coordinate**

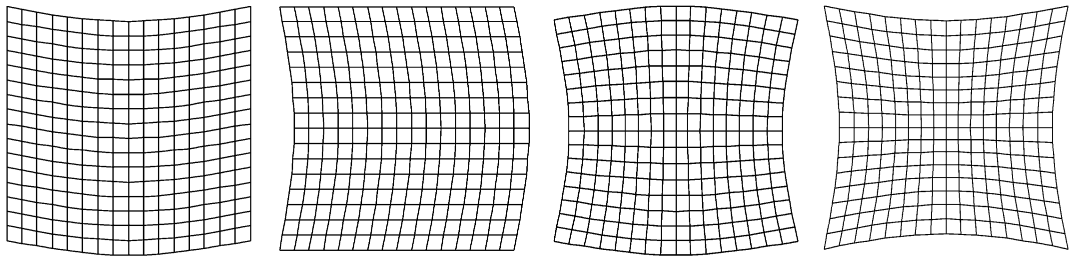
B.C.		1	2	3	4	5	6	7	8	9	10
S1-C	$\Omega a/\pi$	0.5	0.5	1.5	0.5	1.5	2.5	0.5	1.5	2.5	0.5
	Tb/π	0.7255	1.2363	0.5946	1.9876	1.6561	0.5557	2.6131	2.3541	1.6381	3.0163
	Zb/π	0.4630	1.9173	3.6350i	3.4704	2.2270i	6.2473i	4.6886	2.1699	5.5521i	5.4612
	γ	0.8811	1.3336	1.6135	2.0496	2.2345	2.5610	2.6605	2.7914	2.9889	3.0575
S2-C	$\Omega a/\pi$	0.5	0.5	1.5	0.5	1.5	0.5	2.5	1.5	1.5	2.5
	Tb/π	0.7961	1.4294	1.1499	1.8548	2.0238	2.5053	1.1034	2.3081	2.8106	2.1572
	Zb/π	0.7653	2.3353	3.1411i	3.2063	0.5349i	4.4807	5.9908i	1.9919	3.5850	4.9010i
	γ	0.9401	1.5143	1.8900	1.9210	2.5191	2.5547	2.7327	2.7527	3.1858	3.3021
C-C	$\Omega a/\pi$	0.5	0.5	1.5	0.5	1.5	2.5	0.5	1.5	1.5	2.5
	Tb/π	1.1824	1.5860	1.3494	2.1666	2.0245	1.2261	2.9394	2.8070	2.8711	2.3139
	Zb/π	1.7961	2.6619	2.8539i	3.8224	0.5249i	5.9078i	5.3143	3.5753	3.7472	4.6476i
	γ	1.2838	1.6630	2.0176	2.2235	2.5197	2.7845	2.9816	3.1826	3.2393	3.4065
S1-F	$\Omega a/\pi$	0.5	0.5	1.5	0.5	1.5	0.5	2.5	1.5	0.5	2.5
	Tb/π	0.3265i	0.7974	0.6439i	1.4075	0.9344	1.8642	1.0085i	1.9217	2.5072	0.7788
	Zb/π	1.4043i	0.7700	3.9833i	2.2890	3.3791i	3.2250	6.6037i	1.2954i	4.4846	6.1644i
	γ	0.3786	0.9412	1.3548	1.4937	1.7672	1.9301	2.2876	2.4378	2.5566	2.6185
S2-F	$\Omega a/\pi$	0.5	0.5	0.5	1.5	0.5	1.5	2.5	1.5	0.5	1.5
	Tb/π	0.5	0.7543	1.2588	0.5319i	1.9830	1.5	0.9944i	2.0386	2.6087	2.3526
	Zb/π	0.8604i	0.6012	1.9673	3.9258i	3.4613	2.5811i	6.5963i	0.2790i	4.6802	2.1645
	γ	0.7071	0.9049	1.3545	1.4025	2.0451	2.1213	2.2937	2.5310	2.6562	2.7901
C-F	$\Omega a/\pi$	0.5	0.5	1.5	0.5	1.5	2.5	0.5	1.5	1.5	0.5
	Tb/π	0.6998	0.8040	0.4659i	1.6687	1.6356	0.9875i	2.3074	2.3069	2.3562	2.7760
	Zb/π	0.2960	0.7930	3.8967i	2.8311	2.2789i	6.5927i	4.0973	1.9873	2.1780	5.0017
	γ	0.8600	0.9468	1.4258	1.7420	2.2193	2.2967	2.3610	2.7517	2.7932	2.8207
F-F	$\Omega a/\pi$	0.5	0.5	1.5	0.5	0.5	1.5	2.5	0.5	2.5	1.5
	Tb/π	0.4165i	0.6200	0.8206i	1.2162	1.5271	0.7736	1.1259i	2.2017	0.7351i	1.9047
	Zb/π	1.4843i	0.5251i	4.0939i	1.8723	2.5401	3.5167i	6.6689i	3.8912	6.4778i	1.3792i
	γ	0.2766	0.7965	1.2556	1.3149	1.6069	1.6878	2.2321	2.2578	2.3895	2.4244

It can be seen from Table 5 that, for the S1S1S1S1 plate, the frequency parameters of DQM are exactly the same as those of the exact solutions for all the digits used for comparisons. So the frequency parameters of DQM for other plates are not presented. In Figs.2 and 3, the first three mode shapes for S1S1S1S1 plates of the exact method are compared with those of DQM. The eigenvalues in Tables 5 and 6 can be used to draw mode shapes, some of which have been shown in Figs.4-6.

V. CONCLUSIONS

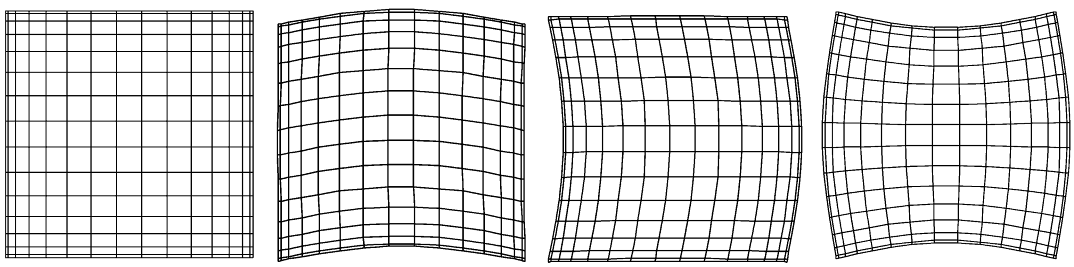
This paper greatly simplified the solving procedures and the forms of the exact solutions for free in-plane vibrations of rectangular plates in Ref.[10]. Moreover, the following improvements are presented:

(1) All possible exact solutions are solved and expressed by 10 sets of distinct eigensolutions.



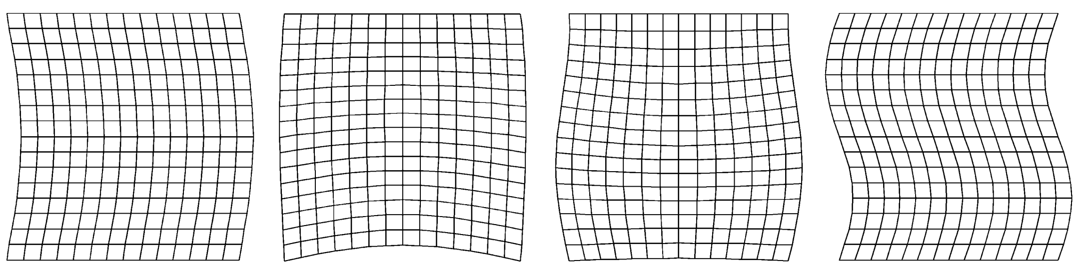
(a) Mode 1 (b) Mode 2 (c) Mode 3 (d) Mode 8

Fig. 2. Modes for an S1S1S1S1 square plate.



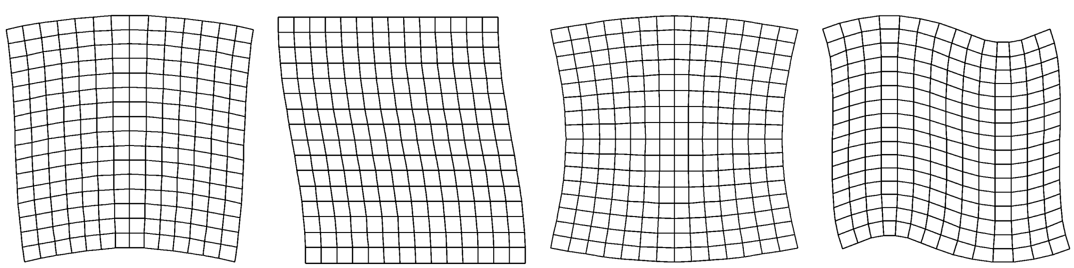
(a) Initial mesh (b) Mode 1 (c) Mode 2 (d) Mode 3

Fig. 3. DQ modes for an S1S1S1S1 square plate.



(a) Mode 2 (b) Mode 3 (c) Mode 4 (d) Mode 5

Fig. 4. Modes for an S1S1S1C square plate.



(a) Mode 1 (b) Mode 2 (c) Mode 3 (d) Mode 4

Fig. 5. Modes for an S1FS1F square plate.

(2) The three cases discussed by Xing and Liu[10] to solve exact solutions are simplified by assuming all imaginary quantities through the solution process as imaginary.

(3) Mathematical proof of the exact solutions is presented.

(4) The unusual phenomena in free in-plane vibrations of rectangular plates due to one of the eigenvalues being zero are revealed.

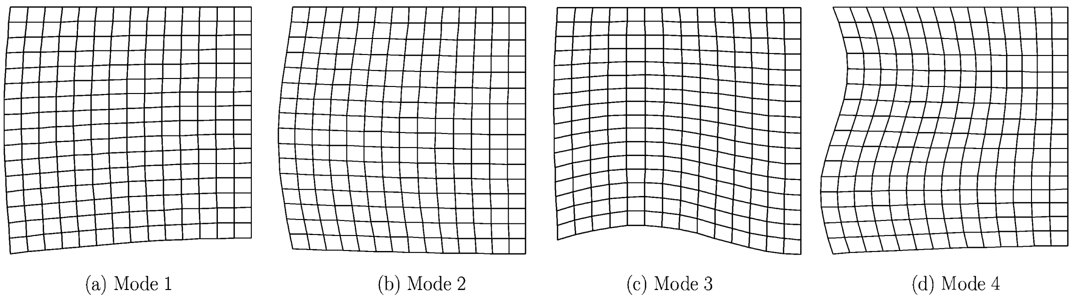


Fig. 6. Modes for an S1S1S2C square plate.

The new exact results can be used as benchmarks for developing new approximate methods. The technique developed in this work is further developed to solve exact solutions for free in-plane vibrations of orthotropic rectangular plates[17].

References

[1] Bercin,A.N., An assessment of the effects of in-plane vibrations on the energy flow between coupled plates. *Journal of Sound and Vibration*, 1996, 191: 661-680.

[2] Lyon,R.H., In-plane contribution to structural noise transmission. *Noise Control Engineering Journal*, 1985, 26: 22-27.

[3] Langley,R.S. and Bercin,A.N., Wave intensity analysis of high frequency vibrations. *Philosophical Transactions of the Royal Society of London A*, 1994, 346: 489-499.

[4] Bardell,N.S., Langley,R.S. and Dunsdon,J.M., On the free in-plane vibration of isotropic plates. *Journal of Sound and Vibration*, 1996, 191: 459-467.

[5] Du,J.T., Li,W.L., Jin,G.Y., Yang,T.J. and Liu,Z.G., An analytical method for the in-plane vibration analysis of rectangular plates with elastically restrained edges. *Journal of Sound and Vibration*, 2007, 306: 908-927.

[6] Gorman,D.J., Exact solutions for the free in-plane vibration of rectangular plates with two opposite edges simply supported. *Journal of Sound and Vibration*, 2006, 294: 131-161.

[7] Singh,A.V. and Muhammad,T., Free in-plane vibration of isotropic non-rectangular plates. *Journal of Sound and Vibration*, 2004, 273: 219-231.

[8] Soedel,W., Vibrations of Shells and Plates, Third Edition. Marcel Dekker, Inc., 2004.

[9] Lord Rayleigh, The Theory of Sound, Vol. 1. Dover, New York, 1894: 395-407.

[10] Xing,Y.F. and Liu,B., Exact solutions for the free in-plane vibrations of rectangular plates. *International Journal of Mechanical Sciences*, 2009, 51: 246-255.

[11] Xing,Y.F. and Liu,B., New exact solutions for free vibrations of rectangular thin plates by symplectic dual method. *Acta Mechanica Sinica*, 2009, 25: 265-270.

[12] Xing,Y.F. and Liu,B., New exact solutions for free vibrations of thin orthotropic rectangular plates. *Composite Structures*, 2009, 89: 567-574.

[13] Gorman,D.J., Accurate analytical type solutions for the free in-plane vibration of clamped and simply supported rectangular plate. *Journal of Sound and Vibration*, 2004, 276: 311-333.

[14] Gorman,D.J., Free in-plane vibration analysis of rectangular plates by the method of superposition. *Journal of Sound and Vibration*, 2004, 272: 831-851.

[15] Gorman,D.J., Free in-plane vibration analysis of rectangular plates with elastic support normal to the boundaries. *Journal of Sound and Vibration*, 2005, 285: 941-966.

[16] Wang,G. and Wereley,N.W., Free in-plane vibration of rectangular plates. *AIAA Journal*, 2002, 40(5): 953-959.

[17] Liu,B. and Xing,Y.F., Comprehensive exact solutions for free in-plane vibrations of orthotropic rectangular plates. *European Journal of Mechanics/A Solids*, 2011, 30: 383-395.

[18] Bhaskar,K. and Sivaram,A., Untruncated infinite series superposition method for accurate flexural analysis of isotropic/orthotropic rectangular plates with arbitrary edge conditions. *Composite Structure*, 2008, 83: 83-92.

[19] Vronay,D.F. and Smith,B.L., Free vibration of circular cylindrical shells of finite length. *AIAA Journal*, 1970, 8(3): 601-603.

[20] Forsberg,K., Influence of boundary conditions on the modal characteristics of thin cylindrical shells. *AIAA Journal*, 1964, 2(11): 2150-2157.

[21] Bert,C.W. and Malik,M., Differential quadrature method in computational mechanics: a review. *Applied Mechanics Review*, 1996, 49: 1-28.

[22] Xing,Y.F. and Liu,B., A differential quadrature finite element method. *International Journal of Applied Mechanics*, 2010, 2(1): 207-227.