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CHARACTERISTIC EQUATIONS AND CLOSED-FORM SOLUTIONS FOR FREE VIBRATIONS OF RECTANGULAR MINDLIN PLATES**

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ABSTRACT The direct separation of variables is used to obtain the closed-form solutions for the free vibrations of rectangular Mindlin plates. Three different characteristic equations are derived by using three different methods. It is found that the deflection can be expressed by means of the four characteristic roots and the two rotations should be expressed by all the six characteristic roots, which is the particularity of Mindlin plate theory. And the closed-form solutions, which satisfy two of the three governing equations and all boundary conditions and are accurate for rectangular plates with moderate thickness, are derived for any combinations of simply supported and clamped edges. The free edges can also be dealt with if the other pair of opposite edges is simply supported. The present results agree well with results published previously by other methods for different aspect ratios and relative thickness.

KEY WORDS Mindlin plate, free vibration, closed-form solution, separation of variables

I. INTRODUCTION

The rectangular plate is an important design element in many branches of modern technology, namely mechanical, aerospace, electronic, marine, optical, nuclear and structural engineering. Thus, the knowledge of its free vibration behaviors is significant to the structural designers. The published works pertaining to the vibrations of such plates are abundant[1,2]. The classical methods focus on thin plates and assume a straight line normal to an undeformed middle surface to remain straight and normal after middle surface deformation. But the neglect of transverse shear deformation and rotary inertia in thick plate analysis results in an overestimation of natural frequency and buckling load and underestimation of bending deflection.

The effect of transverse shear deformation and rotary inertia was considered by Reissner[3] and Mindlin[4] in an effort to develop a more accurate thick plate model. In this first-order shear deformation theory (FSDT), it is assumed that a straight line originally normal to the middle surface remains straight but not generally normal to the middle surface after deformation. This relaxation leads to two rotational degrees of freedom which allow constant transverse shear strain distribution through the thickness. A shear correction factor was derived to account for the deficiency of linear in-plane displacements through the thickness and non-vanishing transverse shear strains on the top and bottom surfaces. The assumption

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of constant strain and linear in-plane displacement limits the application of the Reissner-Mindlin model to the plates with moderate thickness.

Due to the increase of the number of governing equations and independent coordinates, it is more difficult to obtain the exact solutions for the free vibrations of rectangular plate based on FSDT compared with thin plate. For this reason many efforts were devoted to develop approximate solutions with a high level of accuracy based on Mindlin plate theory (MPT). FEM, finite strip method, spline strip method, Rayleigh-Ritz method and collocation methods have been widely used to study the free vibrations of Mindlin rectangular plate. Liew et al.[2] have presented a comprehensive literature survey on the research works up to 1994 on vibrations of thick plates. In the following literature review, more attentions are paid to the works after 1994, but some literatures in Ref.[2] are quoted here to show the representative methods.

The methods of studying the free vibrations of Mindlin rectangular plates include analytical approaches[5–13], as well as numerical procedures[14–26]. Endo and Kimura[5] proposed a two-variable alternative formulation for vibrations of Mindlin plates, in which the bending deflection is regarded as a fundamental variable in place of the rotation angle due to bending. Shimpi and Patel[6] proposed another two-variable refined plate theory which uses the bending component and the shearing component of lateral deflections w as the fundamental variables. Hashemi and Arsanjani[7] studied the exact characteristic equations for some boundary condition combinations wherein at least two opposite edges are simply supported. In fact, for the case with at least one pair of simply-supported opposite edges, an easier solution method based on the inverse method had been presented by Brunelle[8]. Gorman employed the superposition methods to study point supported[9] and completely free[10] Mindlin plates, and the solutions satisfy the governing differential equations exactly and the boundary conditions approximately. Wang[11] presented an explicit formula for the natural frequencies of simply-supported Mindlin plates in terms of the corresponding thin plate frequencies. Xiang[12,13] employed the Levy solution approach associated with the state space technique to derive the analytical solutions for the vibrations of rectangular Mindlin plates.

The most widely used numerical procedure is Rayleigh-Ritz method adopting different kinds of admissible functions. Liew et al.[14–16] adopted two dimensional polynomials and one dimensional Gram-Schmidt polynomials as the admissible functions of the plate, and obtained excellent results. Cheung and Zhou[17] studied the vibrations of moderately thick rectangular Mindlin plates in terms of a set of static Timoshenko beam functions made up of the static solutions of a Timoshenko beam under a series of sinusoidal distributed loads. Shen et al.[18] developed a new set of admissible functions for the free vibration analyses of moderately thick plates with four free edges, and the admissible functions satisfy both geometrical and natural boundary conditions. Liu and Liew[19] developed a two dimensional differential quadrature element method (DQEM) for the free vibrations of thick plates, and a semi-analytical DQEM was given by Malekzadeh et al.[20]. Hou et al.[21] gave a DSC-Ritz method which takes the advantage of both the local bases of the discrete singular convolution (DSC) algorithm and the pb-2 Ritz boundary functions to arrive at a new approach. Diaz-Contreras and Nomura[22] derived numerical Green's functions by using the eigenfunction expansion method, and Sakiyama and Huang[23] presented a Green function method for analyzing the free vibration of thin and moderately thick rectangular plates with arbitrary variable thickness. Lee and Kim[24] gave an iterative method in which Mindlin plate characteristic functions were derived in general forms by the Kantorovich method[25] initially starting with Timoshenko beam functions consistent with the boundary conditions of the plate, and confirmed that the iteration method is superior to the Rayleigh-Ritz analysis or the FEM analysis in accuracy and computational efficiency. Ma and Ang[26] studied the free vibrations of Mindlin plates based on the relative displacement plate element.

To date, the exact solutions for the free vibrations have been obtained for the thick rectangular plates with four simply supported edges[27,28] and with at least two simply-supported opposite edges[7]. It is well-known that the characteristic equations are the most important for finding exact solutions by means of the direct separation of variables, and there are few literatures pertaining to this subject. It is noteworthy that the closed form solutions are valuable in studying the effects of material and structural parameters on the mechanical properties, but the closed-form solution are available in the literature only for the plates with at least two opposite edges simply supported.

In this context, the objective of present work is to study the characteristic equations, to investigate the reasons why the numbers of characteristic roots employed to determine the deflection and the rotations are different as in literature, and to obtain the closed-form solutions for the free vibrations of rectangular Mindlin plates with any combinations of simply supported and clamped edges.

II. GOVERNING EQUATIONS

Consider a thick rectangular plate of length a, width b and uniform thickness h, oriented so that its undeformed middle surface contains the x and y axes of a Cartesian coordinate system (x, y, z), as shown in Fig.1. Three fundamental variables are used to express the displacements in x, y and z directions in MPT, which are

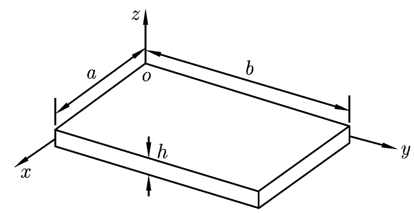

Fig. 1 The coordinates of the rectangular plate.

$$u = -z\psi_x(x, y, z, t), \quad v = -z\psi_y(x, y, z, t), \quad w = w(x, y, z, t) \tag{1}$$

where t is the time, w the transverse deflection of the middle surface, ψ_x and ψ_y are the rotations of a normal line due to plate bending. The resultant bending moments M_x and M_y, the twisting moments M_{xy}, and the transverse shear forces Q_x and Q_y can be obtained by integrating the stresses and the moment of stresses through the thickness of the plate, namely

$$M_x = -D\left(\frac{\partial\psi_x}{\partial x} + \nu\frac{\partial\psi_y}{\partial y}\right), \; M_y = -D\left(\frac{\partial\psi_y}{\partial y} + \nu\frac{\partial\psi_x}{\partial x}\right) \tag{2a}$$

$$M_{xy} = -\frac{1}{2}(1-\nu)D\left(\frac{\partial\psi_x}{\partial y} + \frac{\partial\psi_y}{\partial x}\right) \tag{2b}$$

$$Q_x = C\left(\frac{\partial w}{\partial x} - \psi_x\right), \; Q_y = C\left(\frac{\partial w}{\partial y} - \psi_y\right) \tag{2c}$$

where ν, $D = Eh^3/[12(1-\nu^2)]$ and $C = \kappa Gh$ are Poisson's ratio, the flexural rigidity and the shear rigidity respectively, here κ is the shear correction factor. $G = E/[2(1+\nu)]$ is the shear modulus.

The governing equations for the free vibrations are given by

$$-\frac{\partial M_x}{\partial x} - \frac{\partial M_{xy}}{\partial y} + Q_x - \rho I\frac{\partial^2\psi_x}{\partial t^2} = 0 \tag{3a}$$

$$-\frac{\partial M_{xy}}{\partial x} - \frac{\partial M_y}{\partial y} + Q_y - \rho I\frac{\partial^2\psi_y}{\partial t^2} = 0 \tag{3b}$$

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} - \rho h\frac{\partial^2 w}{\partial t^2} = 0 \tag{3c}$$

where $I = h^3/12$ is moment of inertia. For harmonic normal vibration, it is assumed that

$$\psi_x = \Psi_x(x, y)\,\mathrm{e}^{\mathrm{i}\omega t}, \quad \psi_y = \Psi_y(x, y)\,\mathrm{e}^{\mathrm{i}\omega t}, \quad w = W(x, y)\,\mathrm{e}^{\mathrm{i}\omega t} \tag{4}$$

Substitution of Eqs.(4) into Eqs.(2) and the resulting Eqs.(2) into Eqs.(3) lead to the governing equations in terms of displacements as

$$\frac{\partial^2\Psi_x}{\partial x^2} + \nu_1\frac{\partial^2\Psi_x}{\partial y^2} + \nu_2\frac{\partial^2\Psi_y}{\partial x\partial y} + \frac{C}{D}\left(\frac{\partial W}{\partial x} - \Psi_x\right) + \frac{\gamma^4}{D}\Psi_x = 0 \tag{5a}$$

$$\frac{\partial^2\Psi_y}{\partial y^2} + \nu_1\frac{\partial^2\Psi_y}{\partial x^2} + \nu_2\frac{\partial^2\Psi_x}{\partial x\partial y} + \frac{C}{D}\left(\frac{\partial W}{\partial y} - \Psi_y\right) + \frac{\gamma^4}{D}\Psi_y = 0 \tag{5b}$$

$$\frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 W}{\partial y^2} - \left(\frac{\partial\Psi_x}{\partial x} + \frac{\partial\Psi_y}{\partial y}\right) + \frac{\beta^4}{C}W = 0 \tag{5c}$$

where

$$\nu_1 = \frac{1-\nu}{2}, \; \nu_2 = \frac{1+\nu}{2}, \; \gamma^4 = \omega^2\rho I, \; \beta^4 = \omega^2\rho h \tag{6}$$

III. CHARACTERISTIC EQUATIONS

The characteristic equation method is usually used to obtain the general solutions for homogeneous ordinary equations. Here the authors of this paper use it to obtain the general solutions for vibrating Mindlin plates governed by partial differential equations. The most important step of using the direct separation of variables to solve Eqs.(5) is to obtain its characteristic equations first. In what follows, three different methods are used to derive the characteristic equations.

The first is the algebraic method wherein the particular solutions of Eqs.(5) are written as

$$W\left(x,y\right)=p\mathrm{e}^{\mu x}\mathrm{e}^{\lambda y},\quad \Psi_x\left(x,y\right)=q\mathrm{e}^{\mu x}\mathrm{e}^{\lambda y},\quad \Psi_y\left(x,y\right)=r\mathrm{e}^{\mu x}\mathrm{e}^{\lambda y} \tag{7}$$

where μ and λ are the eigenvalues with respect to coordinates x and y respectively. Substituting Eqs.(7) into Eqs.(5) and according to the condition of nontrivial solution for q, q and r, one can obtain a characteristic equation including the sixth power of μ and λ as follows

$$\left[\left(\mu^2+\lambda^2\right)\frac{D}{C}\nu_1+B\right]\times\left[\left(\mu^2+\lambda^2\right)^2+A\left(\mu^2+\lambda^2\right)+B\frac{\beta^4}{D}\right]=0 \tag{8}$$

where

$$A=\frac{\beta^4}{C}+\frac{\gamma^4}{D},\quad B=\frac{\gamma^4}{C}-1$$

Therefore there are six roots for μ and λ respectively by which the general expressions of the deflection $W(x,y)$ and the rotations Ψ_x and Ψ_y may be determined for isotropic and composite materials, and it is not difficult to solve the roots of Eq.(8) .

The second is the eliminating method. To eliminate Ψ_x and Ψ_y from Eqs.(5), one can obtain

$$\nabla^4 W+A\nabla^2 W+B\frac{\beta^4}{D}W=0 \tag{9}$$

Substitution of Eq.(7a) into Eq.(9) leads to a characteristic equation including the fourth power of μ and λ as

$$\left(\mu^2+\lambda^2\right)^2+A\left(\mu^2+\lambda^2\right)+B\frac{\beta^4}{D}=0 \tag{10}$$

It is noteworthy that Eq.(10) is a factor of Eq.(8), which means the roots of Eq.(10) are also the roots of Eq.(8). It follows from Eqs.(9) and (10) that the general expression of deflection $W(x,y)$ can be determined by using the roots of Eq.(10) or only four roots of Eq.(8). This approach is consistent with the solution methods of thin plates.

The third is also the eliminating method. Eliminating $W(x,y)$ from Eqs.(5) results in two independent equations as follows:

$$\nabla^4\Psi+A\nabla^2\Psi+B\frac{\beta^4}{D}\Psi=0 \tag{11}$$

$$\left(\frac{D}{C}\nu_1\nabla^2+B\right)\frac{\partial\Psi_x}{\partial y}=\left(\frac{D}{C}\nu_1\nabla^2+B\right)\frac{\partial\Psi_y}{\partial x} \tag{12}$$

where

$$\Psi=\frac{\partial\Psi_x}{\partial x}+\frac{\partial\Psi_y}{\partial y}$$

Equation (11) can be rewritten as

$$\left(\nabla^4+A\nabla^2+B\right)\frac{\partial\Psi_x}{\partial x}=-\left(\nabla^4+A\nabla^2+B\right)\frac{\partial\Psi_y}{\partial y} \tag{13}$$

To eliminate Ψ_y or Ψ_x from Eqs.(12) and (13), one can obtain

$$\left(\nabla^4+A\nabla^2+B\right)\left(\frac{D}{C}\nu_1\nabla^2+B\right)\nabla^2\Psi_x=0 \tag{14a}$$

or

$$\left(\nabla^4 + A\nabla^2 + B\right)\left(\frac{D}{C}\nu_1\nabla^2 + B\right)\nabla^2\Psi_y = 0 \tag{14b}$$

Substituting Eq.(7b) into Eq.(14a) or Eq.(7c) into Eq.(14b) yields a characteristic equation

$$\left[\left(\mu^2+\lambda^2\right)^2 + A\left(\mu^2+\lambda^2\right) + B\right] \times \left[\frac{D}{C}\nu_1\left(\mu^2+\lambda^2\right) + B\right] \times \left(\mu^2+\lambda^2\right) = 0 \tag{15}$$

which includes the eighth power of μ and λ, and Eq.(8) is a factor of Eq.(15). Without considering the factor $\left(\mu^2+\lambda^2\right) = 0$, Eq.(8) and Eq.(15) are the same, and Ψ_x and Ψ_y should be determined by using the six roots of Eq.(15). It is worthy to recall that the roots of Eq.(10) or only four roots of Eq.(8) are needed to express the deflection $W(x,y)$. As is well known, the characteristic equations obtained by elimination methods and algebraic method should be the same for simultaneous linear differential equations. But it is apparent from above derivation that the different characteristic equations are obtained by using three different methods, which is the particularity of Mindlin plate theory (MPT).

In Eqs.(8) and (15), there is a common factor as follows

$$\frac{D}{C}\nu_1(\mu^2+\lambda^2) + B = 0 \tag{16}$$

which is identical with the characteristic equation for the free vibration of membrane on elastic foundation. For the free vibrations of thick plates with four edges simply supported, the roots of Eq.(16) have no influence on the two rotations Ψ_x and Ψ_y[8], which means the defection and the rotations can be expressed by the same four characteristic roots. In point of fact, accurate results can also be obtained for moderately thick plates with other boundary combinations through $W(x,y)$, Ψ_x and Ψ_y determined by the four roots of Eq.(10), this will be presented below.

IV. NATURAL MODES

For simplicity and showing a practical approach, the natural mode $W(x,y)$, Ψ_x and Ψ_y are represented only by the four roots of Eq.(10). The roots of characteristic Eq.(10) have the form

$$\mu^2+\lambda^2 = -R_1^2 \quad \text{and} \quad \mu^2+\lambda^2 = R_2^2 \tag{17}$$

where

$$R_1^2 = \frac{\beta^4}{C} + \frac{\gamma^4}{D} + \sqrt{\left(\frac{\beta^4}{C} - \frac{\gamma^4}{D}\right)^2 + \frac{4\beta^4}{D}}, \quad R_2^2 = -\left(\frac{\beta^4}{C} + \frac{\gamma^4}{D}\right) + \sqrt{\left(\frac{\beta^4}{C} - \frac{\gamma^4}{D}\right)^2 + \frac{4\beta^4}{D}}$$

Solving Eqs.(17), one can have

$$\mu_{1,2} = \pm \mathrm{i}\Omega, \quad \mu_{3,4} = \pm\Lambda \tag{18a}$$

$$\lambda_{1,2} = \pm \mathrm{i}T, \quad \lambda_{3,4} = \pm Z \tag{18b}$$

where

$$\Omega = \sqrt{R_1^2+\lambda^2}, \quad \Lambda = \sqrt{R_2^2-\lambda^2} \tag{19a}$$

$$T = \sqrt{R_1^2+\mu^2}, \quad Z = \sqrt{R_2^2-\mu^2} \tag{19b}$$

Thus the general expression in the form of the separation of variables for $W(x,y)$ is

$$W(x,y) = \phi(x)\,\psi(y) \tag{20}$$

where

$$\phi(x) = A_1\cos(\Omega x) + B_1\sin(\Omega x) + C_1\cosh(\Lambda x) + D_1\sinh(\Lambda x) \tag{21}$$

$$\psi(y) = E_1\cos(Ty) + F_1\sin(Ty) + G_1\cosh(Zy) + H_1\sinh(Zy) \tag{22}$$

According to the assumption of MPT, Ψ_x and Ψ_y can be assumed to be

$$\Psi_x(x,y) = g(x)\,\psi(y), \quad \Psi_y(x,y) = \phi(x)\,h(y) \tag{23}$$

where

$$g(x) = q_1\Omega\left[-A_1\sin(\Omega x) + B_1\cos(\Omega x)\right] + q_2\Lambda\left[C_1\sinh(\Lambda x) + D_1\cosh(\Lambda x)\right] \tag{24}$$

$$h(y) = r_1 T\left[-E_1\sin(Ty) + F_1\cos(Ty)\right] + r_2 Z\left[G_1\sinh(Zy) + H_1\cosh(Zy)\right] \tag{25}$$

from which one can see that $\Psi_x = q\partial W/\partial x$, $\Psi_y = r\partial W/\partial y$, and the case $q = r = 1$ corresponds to thin plate theory. The coefficients q and r can be obtained by substituting Eqs.(20) and (23) into Eqs.(5a) and (5b)

$$q_1 = r_1 = \left[1 - \frac{\nu_2 D}{C}\left(R_1^2 - \frac{\beta^4}{C}\right)\right]\left[1 + \frac{\nu_1 D}{C}\left(R_1^2 - \frac{\gamma^4}{\nu_1 D}\right)\right]^{-1} \tag{26a}$$

$$q_2 = r_2 = \left[1 + \frac{\nu_2 D}{C}\left(R_2^2 + \frac{\beta^4}{C}\right)\right]\left[1 - \frac{\nu_1 D}{C}\left(R_2^2 + \frac{\gamma^4}{\nu_1 D}\right)\right]^{-1} \tag{26b}$$

It is apparent that if $C \to \infty$ and $\rho I \to 0$, $q_1 = r_1 = q_2 = r_2 = 1$ and one has the solutions of thin plate problems. From above derivation it can be seen that $W(x,y)$, Ψ_x and Ψ_y defined by Eqs.(20) and (23) satisfy Eqs.(9), (5a) and (5b). And Eq.(9) can be considered as an extension of thin plate theory based on Kirchhoff assumption.

V. BOUNDARY CONDITIONS

In the natural mode functions given in Eqs.(20) and (23), there are altogether eight integral constants, which can be determined by eight boundary conditions. There are three boundary conditions for each edge of rectangular Mindlin plate, but all twelve boundary conditions of rectangular plate with simply-supported and clamped edges can be satisfied by the solutions (20) and (23), because $\Psi_s = 0$ (s denotes the tangent of the edge) is satisfied naturally. As for the free edge, the normal bending moment and total shear force are assumed to be zeros here. This approach is the same as that in thin plates, and is more reasonable and more accurate for practical problems compared with the original approach.

(1) Simply-supported edge

The other two simple support boundary conditions except $\Psi_s = 0$ can be written as

$$W = 0, \quad \frac{\partial \Psi_n}{\partial n} = 0 \tag{27}$$

where n denotes the normal of the edge.

(2) Clamped edge

The other two clamped boundary conditions except $\Psi_s = 0$ have the form

$$W = 0, \quad \Psi_n = 0 \tag{28}$$

(3) Free edge

The three free boundary conditions are

$$M_n = 0, \quad M_{ns} = 0, \quad Q_n = 0 \tag{29}$$

and none of them can be satisfied naturally. Therefore the same approach as in thin plate theory must be used here, that is

$$M_n = 0, \quad Q_n + \frac{\partial M_{ns}}{\partial s} = 0 \tag{30}$$

which can be further rewritten in terms of displacements as

$$\frac{\partial \Psi_n}{\partial n} + \nu\frac{\partial \Psi_s}{\partial s} = 0, \quad \frac{\partial^2 \Psi_n}{\partial n^2} + (1-\nu)\frac{\partial^2 \Psi_n}{\partial s^2} + \frac{\partial^2 \Psi_s}{\partial n \partial s} + \frac{\omega^2 \rho I}{D}\Psi_s = 0 \tag{31}$$

It should be pointed out again that the closed-form solutions can be obtained for any combinations of simply-supported and clamped edges, which is done below. The free edges can also be dealt with if another two opposite edges are simply supported.

VI. EIGENVALUE EQUATIONS AND EIGENFUNCTIONS

Regardless of the two opposite edges being S-C, or others, the eigenvalue equations and the corresponding eigenfunctions can be derived in the same way, so only the case S-C is analyzed here. Assume the edge $x = 0$ is simply supported (S) and the edge $x = a$ is clamped (C), the boundary conditions (S-C) have the form

$$\phi(0) = 0,\ g'(0) = 0; \quad \phi(a) = 0,\ g(a) = 0 \tag{32}$$

Substitution of Eqs.(21) and (24) into Eq.(32) yields

$$A_1 = C_1 = 0 \tag{33}$$

and

$$\begin{bmatrix} \sin(\varOmega a) & \sinh(\varLambda a) \\ q_1 \varOmega \cos(\varOmega a) & q_2 \varLambda \cosh(\varLambda a) \end{bmatrix} \begin{bmatrix} B_1 \\ D_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{34}$$

Due to the conditions of nontrivial solutions, the eigenvalue equation can be obtained

$$q_2 \varLambda \tan(\varOmega a) = q_1 \varOmega \tanh(\varLambda a) \tag{35}$$

Using Eq.(21) in conjunction with Eq.(34a), one can obtain the normal eigenfunction as

$$\phi\,(x) = \sin(\varOmega x) - \frac{\sin(\varOmega a)}{\sinh(\varLambda a)} \sinh(\varLambda x) \tag{36}$$

Similarly, the eigenvalue equations and eigenfunctions can also be obtained for any combination of simply-supported and clamped edges, including the free edge when the other pair of opposite edges is simply supported. Some closed-form eigensolutions are listed in Table 1. For any two eigenvalue equations in Table 1, there are five quantities, i.e., ω, $\varOmega$, $\varLambda$, T and Z, so we need additional three relations. One can obtain a relation from Eq.(19a) as

$$\varOmega^2 + \varLambda^2 = R_1^2 + R_2^2 \tag{37}$$

And substitution of $\mu = \mathrm{i}\varOmega$ into Eq.(19b) results in another two relations

$$T = \sqrt{R_1^2 - \varOmega^2},\ Z = \sqrt{R_2^2 + \varOmega^2} \tag{38}$$

Solving Eqs.(37) and (38) together with the two eigenvalue equations in Table 1, one can obtain the natural frequencies, the eigenvalues and the normal eigenfunctions. And the normal modes are determined by the multiplications of the two normal eigenfunctions, see Eqs.(20) and (23).

VII. NUMERICAL COMPARISONS

Numerical calculations have been carried out for several different combinations of clamped, simply supported and free edge conditions. The frequencies for the plates with four simply-supported edges as well as the cases without simply-supported opposite edges are compared with Liew's[14] in which the *pb*-2 Rayleigh-Ritz method was adopted, as shown in Tables 2-5 wherein the frequency parameter $\lambda = (\omega^2 b^2/\pi^2)\sqrt{\rho h/D}$ are given for relative thickness ratios $h/b = 0.001$, 0.1 and 0.2, aspect ratio $a/b = 0.4$, 0.6, Poisson's ratio $\nu = 0.3$ and $\kappa = 5/6$. MP denotes the present results. It follows from Table 2 that the present results are exact for the plates with four edges simply supported, which means the three governing equations are satisfied exactly.

The results for the plates with two opposite edges simply supported are compared with Hashemi's[7], as shown in Tables 6-10 wherein the frequency parameter $\lambda = (\omega^2 a^2/\pi^2)\sqrt{\rho h/D}$ are presented. The relative thickness ratios $h/a = 0.01$, 0.1 and 0.2 and the aspect ratio $b/a = 0.4$, 0.5 are considered, and the shear correction factor $\kappa = 0.86667$.

It is shown from these comparisons that the present method is accurate enough for practical application in which the relative error for natural frequencies is usually assumed to be less than 5%. But the relative differences of some frequencies seem to be larger for the clamped thick plates with $h/a = 0.5$ or 1/3 (see Table 3). If the free edge is involved, the relative differences also seem to be larger (see Table 9), that is because the different free boundary conditions are used in two methods.

Table 1. **The eigenfunctions and eigenvalue equations**

SSCC	Eigenvalue equations $q_2\Lambda\tan(\Omega a)=q_1\Omega\tanh(\Lambda a),\quad q_2Z\tan(Tb)=q_1T\tanh(Zb)$ Normal eigenfunctions $\phi(x)=\sin(\Omega x)-[\sin(\Omega a)/\sinh(\Lambda a)]\sinh(\Lambda x)$ $\psi(y)=\sin(Ty)-[\sin(Tb)/\sinh(Zb)]\sinh(Zy)$
SCCC	Eigenvalue equations $q_2\Lambda\tan(\Omega a)=q_1\Omega\tanh(\Lambda a)$ $2(q_2Z)(q_1T)[\sinh^{-1}(Zb)-\cos(Tb)\coth(Zb)]+[(q_2Z)^2-(q_1T)^2]\sin(Tb)=0$ Normal eigenfunctions $\phi(x)=\sin(\Omega x)-[\sin(\Omega a)/\sinh(\Lambda a)]\sinh(\Lambda x)$ $\psi(y)=-\cos(Ty)+\theta_1\Pi\sin(Ty)+\cosh(Zy)-\Pi\sinh(Zy)$ $\Pi=[\cos(Ta)-\cosh(Za)]/[\theta_1\sin(Ta)-\sinh(Za)],\quad \theta_1=q_2Z/(q_1T)$
CCCC	Eigenvalue equations $2(q_2\Lambda)(q_1\Omega)[\sinh^{-1}(\Lambda a)-\cos(\Omega a)\coth(\Lambda a)]+[(q_2\Lambda)^2-(q_1\Omega)^2]\sin(\Omega a)=0$ $2(q_2Z)(q_1T)[\sinh^{-1}(Zb)-\cos(Tb)\coth(Zb)]+[(q_2Z)^2-(q_1T)^2]\sin(Tb)=0$ Normal eigenfunctions $\phi(x)=-\cos(\Omega x)+\theta_2\Xi\sin(\Omega x)+\cosh(\Lambda x)-\Xi\sinh(\Lambda x)$ $\Xi=[\cos(\Omega a)-\cosh(\Lambda a)]/[\theta_2\sin(\Omega a)-\sinh(\Lambda a)],\quad \theta_2=q_2\Lambda/(q_1\Omega)$ $\psi(y)=-\cos(Ty)+\theta_1\Pi\sin(Ty)+\cosh(Zy)-\Pi\sinh(Zy)$
SSSF	Eigenvalue equations $(\beta_2/\alpha_2)\tan(Tb)=(\beta_1/\alpha_1)\tanh(Zb)$ $\alpha_1=q_1T^2+\nu q_1\Omega^2,\ \alpha_2=q_2Z^2-\nu q_1\Omega^2$ $\beta_1=q_1T[T^2+(1-\nu)\Omega^2-\gamma^4/D]+q_1T\Omega^2$ $\beta_2=q_2Z[Z^2-(1-\nu)\Omega^2+\gamma^4/D]-q_1Z\Omega^2$ Normal eigenfunctions $\psi(y)=\sin(Ty)+\{q_2\alpha_1\sin(Tb)/[q_1\alpha_2\sinh(Zb)]\}\sinh(Zy)$
SCSF	Eigenvalue equations $(\alpha_2\beta_2+\theta_1\alpha_1\beta_1)\sinh^{-1}(Zb)+(\theta_1\alpha_2\beta_1+\alpha_1\beta_2)\cos(Tb)\coth(Zb)=(\theta_1\alpha_1\beta_2-\alpha_2\beta_1)\sin(Tb)$ Normal eigenfunctions $\psi(y)=\chi_1\cos(Ty)-\theta_1\sin(Ty)-\theta_1\cosh(Zy)+\sinh(Zy)$ $\chi_1=[\alpha_2\sinh(Zb)+\theta_1\alpha_1\sin(Tb)]/[\alpha_2\cosh(Zb)+\alpha_1\cos(Tb)]$
SFSF	Eigenvalue equations $2\alpha_1\beta_1\alpha_2\beta_2[\sinh^{-1}(Zb)-\coth(Zb)\cos(Tb)]=(\alpha_2^2\beta_1^2-\alpha_1^2\beta_2^2)\sin(Tb)$ Normal eigenfunctions $\psi(y)=-k_1\gamma_1\cos(Ty)+k_2\sin(Ty)-\gamma_1\cosh(Zy)+\sinh(Zy),\quad k_1=\alpha_2/\alpha_1$ $\gamma_1=[-\alpha_1k_2\sin(Tb)+\alpha_2\sinh(Zb)]/[-\alpha_2\cos(Tb)+\alpha_2\cosh(Zb)],\quad k_2=\beta_2/\beta_1$

VIII. CONCLUSIONS

The characteristic equations for the free vibrations of rectangular Mindlin plate were investigated extensively. Simple but useful closed-form solutions were obtained and can be used to predict the frequencies for the rectangular plates with any combinations of simply supported and clamped edges. The free edge can also be dealt with by means of present approach when the other two opposite edges are simply supported.

The present results agree well with available exact results and the approximate results by Rayliegh-Ritz method for different aspect ratios and relative thickness, which validates the correctness and the practicableness of the present method and solutions.

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Table 2. **The first eight frequencies** $\lambda = (\omega^2 b^2/\pi^2)\sqrt{\rho h/D}$ **for case SSSS**

a/b	h/b		1	2	3	4	5	6	7	8
0.4	0.001	Liew	7.2500	10.250	15.250	22.249	25.999	26.999	31.248	33.998
		MP	7.2499	10.250	15.250	22.249	25.999	28.998	31.248	33.998
	0.10	Liew	6.4733	8.8043	12.370	16.845	19.050	20.732	21.952	23.399
		MP	6.4733	8.8043	12.370	16.845	19.050	20.732	21.952	23.399
	0.20	Liew	5.1831	6.7212	8.9137	11.487	12.703	13.614	14.267	15.034
		MP	5.1831	6.7212	8.9137	11.487	12.703	13.614	14.267	15.033
0.6	0.001	Liew	3.7777	6.7777	11.778	12.111	16.111	18.777	20.110	25.598
		MP	3.7777	6.7777	11.778	12.111	15.111	18.777	20.110	25.999
	0.10	Liew	3.5465	6.0909	9.9324	10.174	12.275	14.690	15.532	19.050
		MP	3.5465	6.0909	9.9324	10.174	12.275	14.690	15.532	19.050
	0.20	Liew	3.0688	4.9202	7.4327	7.5825	8.8576	10.268	10.748	12.703
		MP	3.0688	4.9202	7.4327	7.5825	8.8576	10.268	10.748	12.703

Table 3. **The first eight frequencies** $\lambda = (\omega^2 b^2/\pi^2)\sqrt{\rho h/D}$ **for case CCCC**

a/b	h/b		1	2	3	4	5	6	7	8
0.4	0.001	Liew	14.972	17.608	22.427	29.553	38.951	39.943	42.671	47.349
		MP	14.910	17.445	22.228	29.374	38.795	39.927	42.601	47.225
	0.10	Liew	10.702	12.352	15.257	19.195	22.627	23.861	23.972	26.198
		MP	10.544	11.921	14.702	18.657	22.530	23.393	23.646	25.643
	0.20	Liew	6.9109	7.9963	9.7716	12.028	13.128	14.013	14.586	15.384
		MP	6.6813	7.5362	9.3262	11.687	13.060	13.817	14.339	15.102
0.6	0.001	Liew	7.2864	10.488	16.018	18.340	21.362	23.751	26.541	33.606
		MP	7.1954	10.352	15.914	18.308	21.274	23.680	26.422	33.540
	0.10	Liew	6.0364	8.3527	12.010	13.049	14.848	16.560	17.764	21.542
		MP	5.8801	8.0773	11.747	12.951	14.560	16.336	17.342	21.124
	0.20	Liew	4.4084	5.9199	8.1258	8.4053	9.5518	10.710	11.280	13.100
		MP	4.1896	5.6064	7.8760	8.3001	9.2956	10.534	10.977	13.018

Table 4. **The first eight frequencies** $\lambda = (\omega^2 b^2/\pi^2)\sqrt{\rho h/D}$ **for case CCCS**

a/b	h/b		1	2	3	4	5	6	7	8
0.4	0.001	Liew	14.842	17.107	21.394	27.930	36.731	39.868	42.376	46.706
		MP	14.817	17.035	21.298	27.839	36.652	39.859	42.344	46.649
	0.10	Liew	10.608	12.048	14.763	18.612	22.596	23.278	23.962	25.998
		MP	10.514	11.749	14.378	18.224	22.521	22.935	23.608	25.562
	0.20	Liew	6.8374	7.8352	9.5938	11.887	13.119	13.985	14.495	15.341
		MP	6.6743	7.4915	9.2573	11.631	13.059	13.815	14.313	15.100
0.6	0.001	Liew	7.0598	9.7309	14.663	18.217	20.899	21.821	25.593	31.114
		MP	7.0238	9.6656	14.608	18.203	20.859	21.782	25.537	31.081
	0.10	Liew	5.8624	7.8731	11.342	12.982	14.629	15.845	17.400	21.001
		MP	5.7865	7.7089	11.168	12.919	14.436	15.690	17.105	20.808
	0.20	Liew	4.2822	5.6866	7.8996	8.3729	9.4678	10.546	11.175	13.089
		MP	4.1535	5.4718	7.7247	8.2931	9.2718	10.424	10.945	13.017

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Table 5. **The first eight frequencies $\lambda = (\omega^2 b^2/\pi^2)\sqrt{\rho h/D}$ for case CCSS**

a/b	h/b		1	2	3	4	5	6	7	8
0.4	0.001	Liew	10.669	13.524	18.507	25.641	32.579	34.888	35.436	40.279
		MP	10.651	13.484	18.462	25.603	32.573	34.858	35.416	40.247
	0.10	Liew	8.5213	10.504	13.752	17.978	20.950	22.412	22.883	24.818
		MP	8.4681	10.359	13.559	17.778	20.905	22.265	22.694	24.566
	0.20	Liew	5.9939	7.3080	9.3061	11.736	12.973	13.846	14.414	15.222
		MP	5.9114	7.1337	9.1261	11.590	12.926	13.729	14.304	15.070
0.6	0.001	Liew	5.3408	8.4789	13.779	15.049	18.061	21.167	23.164	30.324
		MP	5.3187	8.4489	13.756	15.0141	18.039	21.151	23.135	30.323
	0.10	Liew	4.7158	7.1714	10.948	11.640	13.560	15.620	16.633	20.346
		MP	4.6735	7.0916	10.861	11.602	13.449	15.536	16.464	20.314
	0.20	Liew	3.7121	5.3972	7.7639	8.0252	9.2073	10.483	11.006	12.892
		MP	3.6399	5.2810	7.6640	7.9724	9.0878	10.405	10.866	12.857

Table 6. **The first eight frequencies $\lambda = (\omega^2 a^2/\pi^2)\sqrt{\rho h/D}$ for case SCSS**

b/a	h/a		1	2	3	4	5	6	7	8
0.4	0.01	HSH	10.500	12.961	17.397	23.919	32.265	32.515	34.858	39.283
		MP	10.500	12.961	17.396	23.918	32.265	32.514	34.857	39.282
	0.10	HSH	8.4826	10.269	13.354	17.528	21.125	22.483	22.532	24.876
		MP	8.4625	10.207	13.261	17.426	21.095	22.387	22.433	24.710
	0.20	HSH	6.0097	7.2551	9.2573	11.755	13.146	13.204	14.527	15.400
		MP	5.9624	7.1522	9.1523	11.673	13.106	13.917	14.469	15.279
0.5	0.001	HSH	7.0113	9.5615	14.163	20.857	21.014	23.644	28.163	29.599
		MP	7.0113	9.5615	14.163	20.857	21.014	23.643	28.163	29.598
	0.10	HSH	6.0266	8.0241	11.417	15.318	15.888	16.933	19.615	21.090
		MP	6.0142	7.9875	11.364	15.295	15.832	16.861	19.498	21.038
	0.20	HSH	4.5652	5.9904	8.2221	10.104	10.915	11.120	12.729	13.835
		MP	4.5291	5.9186	8.1550	10.068	10.866	11.033	12.627	13.801

Table 7. **The first eight frequencies $\lambda = (\omega^2 a^2/\pi^2)\sqrt{\rho h/D}$ for case SCSC**

b/a	h/a		1	2	3	4	5	6	7	8
0.4	0.01	HSH	14.671	16.604	20.370	26.281	34.399	39.387	41.654	44.669
		MP	14.671	16.604	20.369	26.278	34.397	39.387	41.653	44.666
	0.10	HSH	10.630	11.882	14.408	18.177	22.838	22.881	24.039	26.101
		MP	10.583	11.738	14.194	17.947	22.670	22.782	23.850	25.776
	0.20	HSH	6.8780	7.7803	9.5294	11.892	13.307	14.164	14.600	15.524
		MP	6.7677	7.5492	9.3025	11.721	13.253	14.012	14.480	15.313
0.5	0.001	HSH	9.6220	11.692	15.777	22.076	25.573	27.891	30.546	31.971
		MP	9.6220	11.691	15.777	22.076	25.573	27.891	30.545	31.969
	0.10	HSH	7.6190	9.1229	12.072	16.267	16.898	18.261	20.627	21.314
		MP	7.5891	9.0339	11.948	16.193	16.854	18.117	20.389	21.199
	0.20	HSH	5.2817	6.3805	8.4047	10.409	11.001	11.365	12.901	13.878
		MP	5.2017	6.2230	8.2613	10.350	10.898	11.209	12.706	13.809

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Table 8. **The first eight frequencies** $\lambda = (\omega^2 a^2/\pi^2)\sqrt{\rho h/D}$ **for case SSSF**

b/a	h/a		1	2	3	4	5	6	7	8
0.4	0.01	HSH	1.8988	5.1067	10.123	11.129	14.872	17.061	20.499	25.939
		MP	1.9042	5.1154	10.135	11.146	14.918	17.077	20.563	25.959
	0.10	HSH	1.7974	4.6521	8.6782	9.4467	12.011	13.594	15.667	19.111
		MP	1.8326	4.6666	8.6453	9.5349	12.194	13.506	15.838	18.968
	0.20	HSH	1.6233	3.8806	6.6486	7.1150	8.6527	9.6732	10.778	12.807
		MP	1.6643	3.8626	6.5793	7.2270	8.8720	9.5750	10.992	12.696
0.5	0.001	HSH	1.6310	4.7255	7.6047	9.7043	11.196	16.597	16.632	21.395
		MP	1.6342	4.7310	7.6175	9.7125	11.226	16.635	16.643	21.410
	0.10	HSH	1.5609	4.3454	6.7249	8.3812	9.4200	13.207	13.331	16.450
		MP	1.5813	4.3491	6.7973	8.3488	9.5409	13.253	13.292	16.512
	0.20	HSH	1.4321	3.6642	5.3499	6.4639	7.0900	9.4034	9.5236	11.200
		MP	1.4531	3.6399	5.4357	6.4002	7.2168	9.4366	9.4918	11.314

Table 9. **The first eight frequencies** $\lambda = (\omega^2 a^2/\pi^2)\sqrt{\rho h/D}$ **for case SFSF**

b/a	h/a		1	2	3	4	5	6	7	8
0.4	0.01	HSH	0.9605	3.3805	3.8819	7.5662	8.7914	13.118	15.681	15.742
		MP	0.9998	3.4041	3.9972	7.6073	8.9857	13.171	15.776	15.955
	0.10	HSH	0.9429	3.0030	3.6322	6.4932	7.6926	10.680	12.699	12.786
		MP	0.9829	3.1735	3.7493	6.6573	7.8808	10.738	12.937	12.943
	0.20	HSH	0.9001	2.4525	3.1407	5.0418	6.0226	7.8084	9.1560	9.3462
		MP	0.9373	2.7039	3.2359	5.2146	6.1567	7.8467	9.3119	9.5083
0.5	0.001	HSH	0.9633	2.7721	3.8986	6.5052	8.8243	10.642	11.776	14.774
		MP	0.9998	2.7869	3.9972	6.5301	8.9857	10.669	11.809	14.844
	0.10	HSH	0.9454	2.5301	3.6474	5.7283	7.7223	9.0776	9.8099	11.857
		MP	0.9829	2.6343	3.7493	5.8125	7.8808	9.2098	9.8092	12.131
	0.20	HSH	0.9018	2.1609	3.1507	4.5941	6.0407	6.9723	7.3278	8.5746
		MP	0.9373	2.3054	3.2359	4.6525	6.1567	7.1136	7.2863	8.8670

Table 10. **The first eight frequencies** $\lambda = (\omega^2 a^2/\pi^2)\sqrt{\rho h/D}$ **for case SCSF**

b/a	h/a		1	2	3	4	5	6	7	8
0.4	0.01	HSH	3.0930	5.8619	10.651	15.067	17.455	18.386	23.602	26.247
		MP	3.1004	5.8756	10.668	15.085	17.476	18.440	23.681	26.271
	0.10	HSH	2.8267	5.1580	8.9406	11.392	13.502	13.738	16.728	19.193
		MP	2.8591	5.1655	8.8858	11.458	13.623	13.628	16.809	19.025
	0.20	HSH	2.3922	4.1482	6.7486	7.5552	8.9496	9.7151	10.964	12.827
		MP	2.3869	4.0817	6.6419	7.6478	9.1149	9.5927	11.100	12.700
0.5	0.001	HSH	2.3051	5.1274	9.9766	10.070	13.322	16.831	18.419	25.350
		MP	2.3100	5.1355	9.9870	10.084	13.358	16.845	18.467	25.401
	0.10	HSH	2.1466	4.6175	8.2106	8.5189	10.496	13.405	13.935	18.105
		MP	2.1709	4.6178	8.2694	8.4740	10.586	13.312	13.965	18.138
	0.20	HSH	1.8735	3.8045	5.8641	6.5144	7.4052	9.5443	9.5810	11.606
		MP	1.8768	3.7560	5.9245	6.4315	7.4747	9.4451	9.6001	11.662

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